QUANTUM THEORY OF RADIATION AND INTERACTION WITH MATTER

1) ABELIAN GAUGE THEORY (QED)

2) QUANTIZATION OF PHOTON FIELD

3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON - RELATIVISTIC)

1) ABELIAN GAUGE THEORY (QED)

⇒ LOCAL PHASE TRANSFORMATION

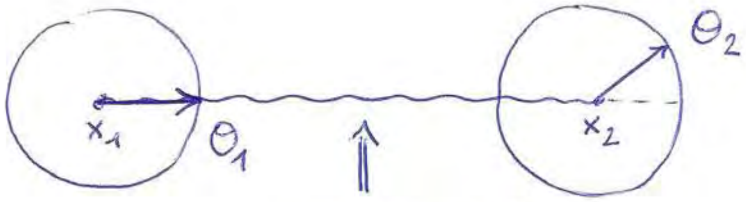
↳ FERMION (SPIN 1/2)

$$\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)$$

↳ LOCAL PHASE TF. $U(1)$ GROUP

$$\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)}\,\psi(x)$$

x → DEPENDS ON SPACE-TIME POINT



$$\Delta\Theta = \Theta_2 - \Theta_1$$

⇑ SIGNAL WHICH COMMUNICATES PHASE DIFFERENCE ("PHOTON")

↳ TRANSFORMATION OF $\mathcal{L}_{DIRAC}$ UNDER LOCAL PHASE TF.

$$\partial_\mu \psi(x) \longrightarrow e^{i\Theta(x)}\left[\partial_\mu \psi(x) + i(\partial_\mu \Theta)\psi\right]$$

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi + i(\partial_\mu\Theta)\bar{\psi}\gamma^\mu\psi$$

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\psi$$

$$\mathcal{L}_{DIRAC} \longrightarrow \mathcal{L}_{DIRAC} - (\partial_\mu\Theta)\overbrace{\bar{\psi}\gamma^\mu\psi}^{\text{VECTOR}}$$

↳ INVARIANCE UNDER LOCAL PHASE TF.

- $$\mathcal{L}_{DIRAC} \xrightarrow{U(1)} \mathcal{L}_{DIRAC} \underbrace{- (\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi}_{\text{EXTRA TERM}}$$

EXTRA TERM $\sim (\partial_\mu \theta)$

DIFFERENCE IN PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- TO MAKE $\mathcal{L}$ INVARIANT UNDER LOCAL PHASE TF $\Longrightarrow$ INTRODUCE VECTOR FIELD (GAUGE FIELD) WHICH COMPENSATES FOR DIFFERENT CHOICES OF PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- INTRODUCE COVARIANT DERIVATIVE

$$\partial_\mu \underset{\text{REPLACE}}{\Longrightarrow} D_\mu = \partial_\mu + i e A_\mu$$

VECTOR FIELD

ELECTRIC CHARGE
COUPLING OF VECTOR FIELD TO DIRAC FIELD

$$A^\mu \xrightarrow{U(1)} A'^\mu$$

CHOOSE A'^μ SUCH THAT TOTAL $\mathcal{L}$ IS INVARIANT UNDER LOCAL PHASE TF.

- $D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$

$$\xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + i(\partial_\mu \theta)\psi + ieA'_\mu \psi \right]$$

$$= e^{i\theta(x)} \left[\partial_\mu \psi + ie\left(A'_\mu + \frac{1}{e}\partial_\mu \theta\right)\psi \right]$$

CHOOSE $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \theta$

$$\Downarrow$$

$$D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + ieA_\mu \psi \right]$$

$$\underline{\underline{D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} D_\mu \psi}}$$

$$\bar{\psi} i \gamma^\mu D_\mu \psi \xrightarrow{U(1)} \bar{\psi} i \gamma^\mu D_\mu \psi$$

INVARIANT UNDER LOCAL U(1) TF !

$\therefore$

$$\boxed{\begin{array}{l} \text{LAGRANGIAN } \mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \\ \text{IS INVARIANT UNDER LOCAL PHASE TF:} \\ \qquad \psi(x) \xrightarrow{U(1)} e^{i\theta(x)}\psi(x) \\ \qquad A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \theta \end{array}}$$

⟹ INTERACTION BETWEEN MATTER & GAUGE FIELD

↳ SYMMERY OF $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ UNDER LOCAL $U(1)$ IS CALLED ABELIAN GAUGE SYMMETRY

↓

REFERS TO GAUGE GROUP $U(1)$

↳ COUPLING TO GAUGE (PHOTON) FIELD

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e(\bar{\psi}\gamma^\mu\psi)A_\mu$$

$$= \mathcal{L}_{DIRAC} + \mathcal{L}_{INT}$$

↑ INTERACTION LAGRANGIAN

$$\boxed{\mathcal{L}_{INT} = -e(\bar{\psi}\gamma^\mu\psi)A_\mu}$$

GRAPHICALLY:

γ (DESCRIBED BY A_μ FIELD).

e IS STRENGTH OF COUPLING BETWEEN FIELDS

⟹ QUANTUM ELECTRODYNAMICS (QED)

↳ INTERACTION BETWEEN MATTER (SPIN 1/2) AND GAUGE FIELDS (PHOTONS) IS DESCRIBED BY

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

- FREE MATTER FIELD ψ

$$\mathcal{L}_{DIRAC} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

- INTERACTION BETWEEN MATTER & GAUGE FIELDS

$$\mathcal{L}_{INT} = -e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

$$\equiv -\underbrace{J^{\mu}_{em}(x)}_{\text{ELECTROMAGNETIC CURRENT}} A_{\mu}$$

$$J^{\mu}_{em} = e\bar{\psi}\gamma^{\mu}\psi$$

- FREE PHOTON FIELD

FIELD TENSOR $\quad \underline{\underline{F^{\mu\nu} \equiv \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}}}$

UNDER $U(1)$ $\quad A^{\mu} \xrightarrow{U(1)} A^{\mu} - \frac{1}{e}\partial^{\mu}\Theta$

$$F^{\mu\nu} \xrightarrow{U(1)} F^{\mu\nu}$$

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

↳ $\mathcal{L}_{QED}$ IS INVARIANT UNDER LOCAL U(1) !

$$\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)} \psi(x)$$

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \Theta$$

U(1) GAUGE SYMMETRY

↳ FIELD EQUATIONS FOR A^μ

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0$$

$\downarrow$ FOR $\phi_r = A_\nu$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = - J^\nu_{em}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - \frac{1}{2} \cdot 2 \cdot F^{\mu\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu_{em}}$$

INHOMOGENEOUS MAXWELL EQ !

↳ EM CURRENT : SOURCE TERM IN MAXWELL EQ.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu_{em}$$

↳ GAUGE INVARIANCE

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \theta$$

FREEDOM TO CONSTRAIN A^μ

POSSIBLE CHOICES OF GAUGE ARE

- $\partial_\mu A^\mu = 0$: LORENZ GAUGE (COVARIANT)
- $\bar{\nabla} . \bar{A} = 0 \quad (A^0 = 0)$: COULOMB GAUGE (NON - COVARIANT)
  ↳ USED FOR FREE FIELDS ($J^\nu_{em} = 0$, ABSENCE OF SOURCES)
- $n_\mu A^\mu = 0$: AXIAL GAUGE
  WITH $n_\mu n^\mu = -1$
  e.g. $n^\mu (0,0,0,1)$

IN FOLLOWING WE WILL OFTEN USE LORENZ GAUGE $\partial_\mu A^\mu = 0$

↓

$$\square A^\nu = J^\nu_{em}$$

$$\partial_\mu A^\mu = 0 \qquad \text{CONSTRAINT}$$

GAUGE INVARIANCE ENSURES THAT RESULTS FOR PHYSICAL OBSERVABLES ARE INVARIANT UNDER CHOICE OF GAUGE

2) QUANTIZATION OF PHOTON FIELD

$\Rightarrow$ PRESENCE OF CONSTRAINTS : ISSUES

$$\Box A^\nu = J^\nu_{em}$$

$$\partial_\nu A^\nu = 0 \qquad \text{(LORENZ GAUGE)}$$

HOW TO QUANTISE A^ν IN PRESENCE OF CONSTRAINT $\partial_\nu A^\nu = 0$.

ISSUE 1 : CANONICAL MOMENTA

$$A^\nu \rightarrow \Pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu}$$

$$\Pi^0 = 0$$

$$\Pi^i = -F^{0i} = -\partial^0 A^i + \partial^i A^0 = \left(-\bar{\nabla} A^0 - \frac{\partial \bar{A}}{\partial t}\right)^i = \bar{E}^i \quad \text{(ELECTRIC FIELD)}$$

FOR A^1, A^2, A^3 OK

FOR $A^0 \Rightarrow$ WE CANNOT IMPOSE COMMUTATION RELATIONS BECAUSE $\Pi^0 = 0$

⇒ GAUGE FIXING

↳ TO SOLVE ABOVE PROBLEMS WHEN QUANTIZING THE ABELIAN GAUGE THEORY CONSIDER ALTERNATIVE $\mathcal{L}$

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ DESCRIBES FREE PHOTON FIELD

$$\mathcal{L}_{em} \longrightarrow \boxed{\mathcal{L}'_{em} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\mathcal{L}_{em}} + \underbrace{- \frac{1}{2} (\partial_\mu A^\mu)^2}_{\mathcal{L}_{GF}}}$$

(GAUGE FIXING)

↳ EQUATIONS OF MOTION

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - F^{\mu\nu} - (\partial_\alpha A^\alpha)\, g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu(\partial_\alpha A^\alpha) = 0$$

$$\Box A^\nu - \partial^\nu(\cancel{\partial_\mu A^\mu}) + \partial^\nu(\cancel{\partial_\alpha A^\alpha}) = 0$$

$$\Box A^\nu = 0$$ ⤳ FIELD EQ. CORRESPONDING WITH $\mathcal{L}'_{em}$ CORRESPOND WITH CHOICE OF LORENZ GAUGE

↳ CANONICAL MOMENTA

$$A^\nu \to \Pi^\nu = - F^{0\nu} - (\partial_\alpha A^\alpha)\, g^{0\nu}$$

$$\begin{cases} \Pi^0 = - \partial_\alpha A^\alpha \\ \Pi^i = - F^{0i} \end{cases}$$

, OK ISSUE 1 IF $\partial_\alpha A^\alpha$ CONSIDERED AS OPERATOR

↳ EQUAL TIME COMMUTATION RELATIONS (ETCR)

TO QUANTIZE THE THEORY, WE IMPOSE

$$\left[A^\mu(\bar{x},t),\, A^\nu(\bar{x}',t)\right]_- = 0$$

$$\left[\Pi^\mu(\bar{x},t),\, \Pi^\nu(\bar{x}',t)\right]_- = 0$$

$$\left[A^\mu(\bar{x},t),\, \Pi^\nu(\bar{x}',t)\right]_- = i\, g^{\mu\nu}\, \delta^3(\bar{x}-\bar{x}')$$

WITH $[A,B]_- = AB - BA$ COMMUTATOR

SPIN 1 FIELD DESCRIBES BOSONS (PHOTONS)

A^μ, Π^μ CONSIDERED AS OPERATORS

$$\rightsquigarrow \left[A^\mu(\bar{x},t),\, \Pi^0(\bar{x}',t)\right]_- = i\, g^{\mu 0}\, \delta^3(\bar{x}-\bar{x}')$$

$$= - \left[A^\mu(\bar{x},t),\, \dot{A}^0(\bar{x}',t)\right]_- - \bar{\nabla}_{x'} \cdot \underbrace{\cancel{\left[A^\mu(\bar{x},t),\, \bar{A}(\bar{x}',t)\right]_-}}_{0}$$

$$\rightsquigarrow \quad [A^\mu(\bar{x},t),\ \underbrace{\Pi^i(\bar{x}',t)}_{-\dot{A}^i + \partial^i A^0}]_- = i g^{\mu i}\,\delta^3(\bar{x}-\bar{x}')$$

$$= -[A^\mu(\bar{x},t),\ \dot{A}^i(\bar{x}',t)]_- - \bar{\nabla}^i_{x'} \cdot \underbrace{\cancel{[A^\mu(\bar{x},t),\ A^0(\bar{x}',t)]_-}}_{0}$$

$$\therefore \quad \| \ [A^\mu(\bar{x},t),\ \dot{A}^\nu(\bar{x}',t)]_- = -i g^{\mu\nu}\,\delta^3(\bar{x}-\bar{x}')$$

$\hookrightarrow \mu = 0$: $\boxed{[A^\mu(\bar{x},t),\ \dot{A}^0(\bar{x}',t)]_- = -i\,\delta^3(\bar{x}-\bar{x}')}$

$\hookrightarrow \mu = i$: $\boxed{[A^\mu(\bar{x},t),\ \dot{A}^i(\bar{x}',t)]_- = +i\,\delta^3(\bar{x}-\bar{x}')}$

$\hookrightarrow$ PHOTON PROPAGATOR (IN LORENZ GAUGE)

$$\Box A^\mu = 0$$

$$\underbrace{(-k^2 g^{\mu\nu})}_{\text{INVERSE EXISTS}} A_\nu = 0$$

$$(-k^2 g^{\mu\nu}) \cdot (A\, g_{\nu\lambda} + B\, k_\nu k_\lambda) = g^\mu{}_\lambda$$

$$A = -\frac{1}{k^2} \qquad B = 0$$

$\Rightarrow$ PHOTON PROPAGATOR

$$\sim \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}$$

$\nu \xrightarrow{k} \mu$

⟹ NORMAL MODE EXPANSION OF $A^\mu(\bar{x},t)$

↳ UPON QUANTIZATION

CLASSICAL FIELDS A^μ

ARE RE-INTERPRETED AS FIELD OPERATORS $\hat{A}^\mu$

THAT SATISFY ETCR

(NOTE: FOR SIMPLICITY WE WILL DROP ^ NOTATION
IN QFT THE FIELDS ARE UNDERSTOOD
AS OPERATORS)

↳ POLARIZATION VECTORS:

A NORMAL MODE SOLUTION IS CHARACTERIZED

BY $\sim e^{ik.x}\, \varepsilon^\mu(\bar{k},\lambda) \qquad \lambda = 0,1,2,3$

WITH $k\,(\omega_{\bar{k}}, \underbrace{0,0,|\bar{k}|}_{\bar{k}})$ CONVENIENT CHOICE $\bar{k} = |\bar{k}|\,\bar{e}_z$

$\omega_{\bar{k}} = |\bar{k}|$

$\varepsilon^\mu(\bar{k},\lambda=0) = (1,0,0,0)$ SCALAR POL.

$\left.\begin{aligned}\varepsilon^\mu(\bar{k},\lambda=1) &= (0,1,0,0)\\ \varepsilon^\mu(\bar{k},\lambda=2) &= (0,0,1,0)\end{aligned}\right\}$ TRANSVERSE POL. $0 = \bar{k}\cdot\bar{\varepsilon}(\bar{k},\lambda=\tfrac{1}{2})$

$\varepsilon^\mu(\bar{k},\lambda=3) = (0,0,0,1)$ LONGITUDINAL POL.

NORMALIZATION $\quad \varepsilon^\mu(\bar{k},\lambda)\,\varepsilon^*_\mu(\bar{k},\lambda') = -\zeta_\lambda\,\delta_{\lambda\lambda'}$

$(\zeta_0 \equiv -1,\ \zeta_i \equiv +1)$

↳ NORMAL MODE EXPANSION

$$\boxed{A^\mu(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} \left\{ a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} + a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\, e^{+ik.x} \right\}}$$

$\omega_{\bar{k}} = |\bar{k}|$ PARTICLE WITH MASS 0 (PHOTON)

(→ MOVES WITH SPEED OF LIGHT)

$a(\bar{k},\lambda)$ ANNIHILATES PHOTON WITH MOMENTUM $\bar{k}$ & POLARIZATION λ

$a^+(\bar{k},\lambda)$ CREATES PHOTON " "

⟹ COMMUTATION RELATIONS FOR a, a^+

↳ $$\int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) + \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$$

$$= 2 \frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\, e^{-i\omega_{\bar{k}} t}$$

↳ $$\int d^3\bar{x}\; e^{+i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) - \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$$

$$= 2 \frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\, e^{i\omega_{\bar{k}} t}$$

$$\downarrow \quad \text{USING} \quad \varepsilon^\mu(\bar{k},\lambda)\, \varepsilon^*_\mu(\bar{k},\lambda') = -\zeta_\lambda\, \delta_{\lambda\lambda'}$$

$$a(\bar{k},\lambda) = -\zeta_\lambda \frac{1}{2}\sqrt{2\omega_{\bar{k}}}\, e^{+i\omega_{\bar{k}} t} \int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}}\, \varepsilon^*_\mu(\bar{k},\lambda) \left\{ A^\mu + \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu \right\}$$

$$a^\dagger(\bar{k},\lambda) = -\zeta_\lambda \frac{1}{2}\sqrt{2\omega_{\bar{k}}}\, e^{-i\omega_{\bar{k}} t} \int d^3\bar{x}\, e^{+i\bar{k}.\bar{x}}\, \varepsilon_\mu(\bar{k},\lambda) \left\{ A^\mu - \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu \right\}$$

$$\hookrightarrow \left[a(\bar{k},\lambda), a^\dagger(\bar{k}',\lambda') \right]_-$$

$$= \zeta_\lambda \zeta_{\lambda'} \frac{1}{4} (2\omega_{\bar{k}})^{1/2} (2\omega_{\bar{k}'})^{1/2} e^{i(\omega_{\bar{k}} - \omega_{\bar{k}'})t}\, \varepsilon^*_\mu(\bar{k},\lambda)\, \varepsilon_\nu(\bar{k}',\lambda')$$

$$\cdot \int d^3\bar{x} \int d^3\bar{x}'\, e^{-i\bar{k}.\bar{x}}\, e^{+i\bar{k}'.\bar{x}'}$$

$$\cdot \underbrace{\left[A^\mu(\bar{x},t) + \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t),\ A^\nu(\bar{x}',t) - \frac{i}{\omega_{\bar{k}'}} \dot{A}^\nu(\bar{x}',t) \right]_-}$$

$$-\frac{1}{\omega_{\bar{k}'}} g^{\mu\nu} \delta^3(\bar{x}-\bar{x}') - \frac{1}{\omega_{\bar{k}}} g^{\mu\nu} \delta^3(\bar{x}-\bar{x}')$$

$$= \zeta_\lambda \zeta_{\lambda'} \frac{1}{4} (2\omega_{\bar{k}})^{1/2} (2\omega_{\bar{k}'})^{1/2} (-1) \left(\frac{1}{\omega_{\bar{k}}} + \frac{1}{\omega_{\bar{k}'}} \right) e^{i(\omega_{\bar{k}} - \omega_{\bar{k}'})t}$$

$$\cdot\, \varepsilon^*_\mu(\bar{k},\lambda)\, \varepsilon^\mu(\bar{k}',\lambda') \underbrace{\int d^3\bar{x}\, e^{-i(\bar{k}-\bar{k}').\bar{x}}}_{(2\pi)^3 \delta^3(\bar{k}-\bar{k}')}$$

$$[\, a(\bar{k},\lambda)\,,\, a^{+}(\bar{k}',\lambda')\,]_{-}$$

$$= -\,\zeta_{\lambda}\,\zeta_{\lambda'}\;\frac{1}{4}\,(2\omega_{\bar{k}})\left(\frac{2}{\omega_{\bar{k}}}\right)\cdot(2\pi)^{3}\;\underbrace{\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon^{\mu}(\bar{k},\lambda')}_{-\zeta_{\lambda}\,\delta_{\lambda\lambda'}}$$

$$\downarrow\quad \zeta_{\lambda}^{2} = 1$$

$$= \zeta_{\lambda}\;\delta_{\lambda\lambda'}\;(2\pi)^{3}\;\delta^{3}(\bar{k}-\bar{k}')$$

$$\therefore\quad \boxed{\begin{array}{l} [\, a(\bar{k},\lambda)\,,\, a^{+}(\bar{k}',\lambda')\,]_{-} = \zeta_{\lambda}\,\delta_{\lambda\lambda'}(2\pi)^{3}\,\delta^{3}(\bar{k}-\bar{k}') \\ \text{ANALOGOUSLY} \\ [\, a(\bar{k},\lambda)\,,\, a(\bar{k}',\lambda')\,]_{-} = 0 \\ [\, a^{+}(\bar{k},\lambda),\, a^{+}(\bar{k}',\lambda')\,]_{-} = 0 \end{array}}$$

$$\zeta_{0} = -1,\quad \zeta_{i} = +1 \quad (i = 1,2,3)$$

FOR $\lambda = 1, 2, 3$

$$[\, a(\bar{k},\lambda)\,,\, a^{+}(\bar{k}',\lambda)\,]_{-} = \delta_{\lambda\lambda'}\;(2\pi)^{3}\delta^{3}(\bar{k}-\bar{k}')$$

FOR $\lambda = 0$

$$[\, a(\bar{k},0)\,,\, a^{+}(\bar{k}',0)\,]_{-} = -\;(2\pi)^{3}\;\delta^{3}(\bar{k}-\bar{k}')$$

FOR $\lambda = 1,2,3$ $\longrightarrow$ STANDARD BOSON COMMUTATION RELATIONS

$\lambda = 0$ $\longrightarrow$ SIGN CHANGE

↳ IF WE CONSIDER VACUUM STATE

AS STATE WITHOUT PHOTONS

i.e. $a(\bar{k},\lambda)\,|0\rangle = 0$ $\qquad \lambda = 0, 1, 2, 3$

$\Downarrow$

1 - PHOTON STATE WITH POLARIZATION λ

$$|1\gamma, \lambda\rangle = \int d^3\bar{k}\ f(\bar{k})\ a^{+}(\bar{k},\lambda)\,|0\rangle$$

↳ WAVE PACKET

$$\int d^3\bar{k}\ |f(\bar{k})|^2 < \infty$$

NORMALIZATION OF 1γ STATE

$$\langle 1\gamma, \lambda | 1\gamma, \lambda\rangle$$

$$= \int d^3\bar{k} \int d^3\bar{k}'\ f^{*}(\bar{k})\, f(\bar{k}') \underbrace{\langle 0|\, a(\bar{k},\lambda)\, a^{+}(\bar{k}',\lambda)\,|0\rangle}_{\zeta_\lambda\,(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')}$$

$$= \zeta_\lambda\,(2\pi)^3 \int d^3\bar{k}\ |f(\bar{k})|^2$$

$$= \begin{cases} > 0 & \text{FOR } \lambda = 1,2,3 \\ < 0 & \text{FOR } \lambda = 0 \ \text{☹ NEGATIVE NORM STATE} \end{cases}$$

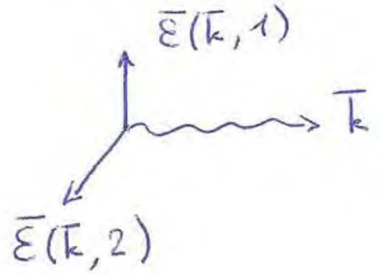
↳ HAMILTONIAN :

$$H = \int \frac{d^3\bar{k}}{(2\pi)^3}\, \omega_{\bar{k}} \left\{ - a^{+}(\bar{k},0)\, a(\bar{k},0) + \sum_{i=1}^{3} a^{+}(\bar{k},i)\, a(\bar{k},i) \right\}$$

STATES WITH $\lambda = 0$ LEAD TO NEGATIVE ENERGY !

↳ PHYSICAL STATES :

FREE MAXWELL FIELD HAS ONLY 2 TRANSVERSE COMPONENTS



↳ $\lambda = 0$, $\lambda = 3$ STATES SHOULD NOT APPEAR IN PHYSICAL STATES UPON QUANTIZATION

$|\Psi\rangle$ PHYSICAL VACUUM (HAS NO TRANSVERSE PHOTONS)

WE REQUIRE

$$\partial_\mu A^{\mu(+)} |\Psi\rangle = 0 \qquad \text{(GUPTA - BLEULER)}$$

(+) STANDS FOR POS. FREQUENCY (ANNIHILATING) PART IN A^μ

NOTE : THIS IS A WEAKER CONDITION THAN CONSIDERING $\partial_\mu \hat{A}^\mu = 0$ AS OPERATOR CONDITION

$$\partial_\mu A^{\mu(+)} | \Psi \rangle = 0$$

$$\downarrow$$

$$\partial_\mu \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda a(\bar{k},\lambda)\, \varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} \quad |\Psi\rangle = 0$$

$$\int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda (-i)\, a(\bar{k},\lambda)\, e^{-ik.x} \quad k_\mu \varepsilon^\mu(\bar{k},\lambda) \,|\Psi\rangle = 0$$

$$\begin{aligned} \Big\downarrow \qquad & k_\mu \varepsilon^\mu(\bar{k},\lambda) \\ & = \omega_{\bar{k}}\, \varepsilon^0(\bar{k},\lambda) - |\bar{k}|\, \varepsilon^3(\bar{k},\lambda) \\ & = \omega_{\bar{k}} (\delta_{\lambda 0} - \delta_{\lambda 3}) \end{aligned}$$

$$\forall \bar{k} : \quad \underline{\underline{\big(a(\bar{k},0) - a(\bar{k},3)\big) |\Psi\rangle = 0.}}$$

$$\Downarrow$$

$$\langle \Psi | a^\dagger(\bar{k},3)\, a(\bar{k},3) | \Psi \rangle = \langle \Psi | a^\dagger(\bar{k},0)\, a(\bar{k},0) | \Psi \rangle$$

- PHYSICAL VACUUM STATE HAS EQUAL NUMBER OF SCALAR ($\lambda = 0$) AND LONGITUDINAL ($\lambda = 3$) PHOTONS

- THE COMBINED ENERGY OF SCALAR & LONGITUDINAL PHOTONS IS ZERO

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3 \bar{k}}{(2\pi)^3} \, \omega_{\bar{k}} \sum_{\lambda=1,2} \langle \Psi | a^{+}(\bar{k},\lambda) \, a(\bar{k},\lambda) | \Psi \rangle$$

- PHYSICAL QUANTITIES ONLY INVOLVE TRANSVERSE PHOTONS ($\lambda = 1, 2$).

- ALTERING THE ALLOWED ADMIXTURES OF SCALAR AND LONGITUDINAL PHOTONS IS EQUIVALENT TO A GAUGE TF. BETWEEN 2 POTENTIALS, BOTH OF WHICH ARE IN LORENZ GAUGE

  ($\rightarrow$ EXERCISE)

3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON-RELATIVISTIC)

- HAMILTONIANS

↳ MANY-BODY SYSTEM (MATTER) $\quad H_M = \sum_{i=1}^{N} \frac{\bar{P}_i^2}{2m_i} + V$

↳ RADIATION FIELD $\quad H_{EM} = \frac{1}{2} \int d^3\bar{x} \left(\bar{E} \cdot \bar{E}^* + \bar{B} \cdot \bar{B}^* \right)$

↳ INTERACTION HAS TO RESPECT GAUGE-INVARIANCE

$$\psi(\bar{x},t) \longrightarrow e^{\frac{ie}{\hbar c}\chi(\bar{x},t)} \psi(\bar{x},t)$$

LOCAL PHASE TRANSFORMATION HAS TO LEAVE THEORY INVARIANT

$$\hat{P}\,\psi(\bar{x},t) = -i\hbar \bar{\nabla} \psi(\bar{x},t)$$

$$\longrightarrow e^{\frac{ie}{\hbar c}\chi(\bar{x},t)} \left[-i\hbar \bar{\nabla}\psi + \frac{e}{c}\left(\bar{\nabla}\chi\right)\psi \right]$$

KINETIC ENERGY TERM CAN BE MADE INVARIANT BY MINIMAL SUBSTITUTION

$$\boxed{\bar{P}_i \Rightarrow \bar{P}_i - \frac{e_i}{c}\bar{A}(x_i)}$$

UNDER GAUGE TRANSFORMATION, $\bar{A}$ TRANSFORMS AS

$$\bar{A} \to \bar{A} + \bar{\nabla}\chi$$

$$\therefore \left(\bar{p} - \frac{e}{c}\bar{A}\right)\psi(\bar{x},t)$$

$$\to e^{\frac{ie}{\hbar c}\chi(\bar{x},t)}\left\{-i\hbar\bar{\nabla} + \cancel{\frac{e}{c}(\bar{\nabla}\chi)} - \frac{e}{c}\bar{A} - \cancel{\frac{e}{c}(\bar{\nabla}\chi)}\right\}\psi$$

$$= e^{\frac{ie}{\hbar c}\chi(\bar{x},t)}\left(\underbrace{-i\hbar\bar{\nabla}}_{\bar{p}} - \frac{e}{c}\bar{A}\right)\psi$$

$\therefore$ HAMILTONIAN IS GAUGE INVARIANT

$\Downarrow$

EXPECTATION VALUES REMAIN UNCHANGED

e.g. $\int d^3\bar{x}\ \psi^*(\bar{x},t)\frac{1}{2m}\left(-i\hbar\bar{\nabla} - \frac{e}{c}\bar{A}\right)^2\psi(\bar{x},t)$

$$\boxed{\begin{aligned} H_{tot} &= \sum_{i=1}^{N}\frac{1}{2m_i}\left(\bar{p}_i - \frac{e_i}{c}\bar{A}_i\right)^2 + V + H_{EM} \\ &= H_M + H_{INT} + H_{EM}\end{aligned}}$$

$\uparrow$ MATTER

$\uparrow$ INTERACTION RADIATION WITH MATTER

↳ INTERACTION HAMILTONIAN

$$\boxed{H_{INT} = \sum_i \left\{ -\frac{e_i}{m_i c}\,\bar{P}_i\cdot\bar{A}(\bar{x}_i) + \frac{e_i^2}{2m_i c^2}\,\bar{A}^2(\bar{x}_i)\right\} \equiv H'_{INT} + H''_{INT}}$$

LINEAR IN $\bar{A}$ QUADRATIC IN $\bar{A}$

(NOTE IN H'_{INT}: $\bar{P}_i\cdot\bar{A} + \bar{A}\cdot\bar{P}_i = 2\,\bar{P}_i\cdot\bar{A}$ (IN COULOMB GAUGE $\bar{\nabla}\cdot\bar{A} = 0$)

- INTERACTION HAMILTONIAN IN SECOND QUANTIZATION

↳ RECALL:
$$\bar{A}(\bar{x}) = \sum_{\bar{k}}\sum_{\sigma=1,2} N_k\,\varepsilon_{\bar{k}\sigma}\left\{\hat{a}_{\bar{k}\sigma}\,e^{i\bar{k}\cdot\bar{x}} + \hat{a}^{+}_{\bar{k}\sigma}\,e^{-i\bar{k}\cdot\bar{x}}\right\}$$

WITH
$$N_k = \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2}$$

↳ INTERACTION OF 1 PARTICLE WITH RADIATION FIELD

$$\hat{H}'_{INT} = -\frac{e}{mc}\sum_{\bar{k}}\sum_{\sigma} N_{\bar{k}}\,\hat{\bar{P}}\cdot\varepsilon_{\bar{k}\sigma}\left\{\hat{a}_{\bar{k}\sigma}\,e^{i\bar{k}\cdot\bar{x}} + \hat{a}^{+}_{\bar{k}\sigma}\,e^{-i\bar{k}\cdot\bar{x}}\right\}$$

$$\hat{H}''_{INT} = \frac{e^2}{2mc^2} \sum_{\bar{k}\sigma} \sum_{\bar{k}'\sigma'} \left(\frac{\hbar c^2}{2L^3}\right) \frac{\bar{\varepsilon}_{\bar{k}\sigma} \cdot \bar{\varepsilon}_{\bar{k}'\sigma'}}{(\omega_k \, \omega_{k'})^{1/2}}$$

$$\cdot \Big\{ \hat{a}_{\bar{k}\sigma} \, \hat{a}_{\bar{k}'\sigma'} \, e^{i(\bar{k}+\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}_{\bar{k}\sigma} \, \hat{a}^{+}_{\bar{k}'\sigma'} \, e^{i(\bar{k}-\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}^{+}_{\bar{k}\sigma} \, \hat{a}_{\bar{k}'\sigma'} \, e^{i(-\bar{k}+\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}^{+}_{\bar{k}\sigma} \, \hat{a}^{+}_{\bar{k}'\sigma'} \, e^{-i(\bar{k}+\bar{k}')\cdot\bar{x}} \Big\}$$

↳ WE WILL CONSIDER $H'_{INT} + H''_{INT}$ AS PERTURBATION TO $H_M + H_{EM}$

UNPERTURBED STATES

$$|\Psi_0\rangle = |\ldots n_i \ldots\rangle_M \otimes |\ldots n_{\bar{k}\sigma} \ldots\rangle_{EM}$$

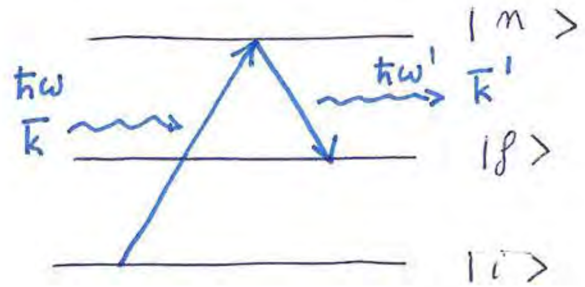
MATTER ↑ RADIATION FIELD ↑

$H'_{INT} + H''_{INT}$ WILL INDUCE TRANSITIONS BETWEEN THESE UNPERTURBED STATES.

↳ EXAMPLES : PHOTONS ARE ABSORBED ($\hat{a}_{\bar{k}\sigma}$)

AND/OR EMITTED ($\hat{a}^{+}_{\bar{k}\sigma}$)

1)

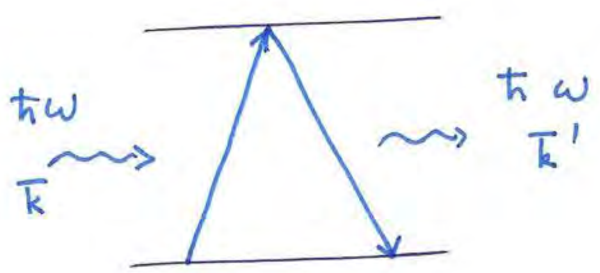


INELASTIC PHOTON SCATTERING

(RAMAN SCATTERING)

$\omega' \neq \omega, \quad \bar{k}' \neq \bar{k}$

2) IF $|i\rangle = |f\rangle$



ELASTIC PHOTON SCATTERING

$\omega' = \omega, \quad |\bar{k}'| = |\bar{k}|$

BUT DIRECTIONS CAN BE DIFFERENT

- **INTERLUDE : TIME - DEPENDENT PERTURBATION THEORY**

↳ QUANTUM SYSTEM DESCRIBED BY $\hat{H}_0$ (UNPERTURBED HAMILT.)

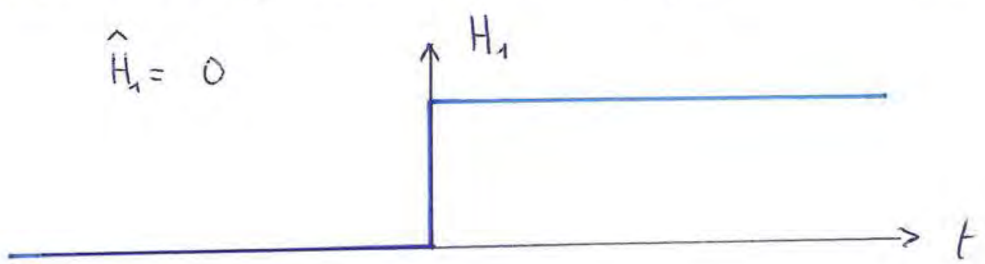
$$\hat{H}_0 \, | \Psi_0(t) \rangle = i\hbar \frac{\partial}{\partial t} | \Psi_0(t) \rangle$$

$$| \Psi_0(t) \rangle = \sum_n a_n \, e^{-\frac{i}{\hbar} E_n t} \, | \psi_n \rangle$$

$$\hat{H}_0 \, | \psi_n \rangle = E_n \, | \psi_n \rangle$$

↑ EIGENSTATES ↑ EIGENVALUES.

↳ AT $t = 0$, PERTURBATION $\hat{H}_1$ IS SWITCHED ON



$\underset{\sim}{t > 0}$: $\hat{H} = \hat{H}_0 + \hat{H}_1$

$$\left(\hat{H}_0 + \hat{H}_1 \right) | \Psi(t) \rangle = i\hbar \frac{\partial}{\partial t} | \Psi(t) \rangle$$

EXPAND $| \Psi(t) \rangle$ IN EIGENSTATES OF $\hat{H}_0$

$$| \Psi(t) \rangle = \sum_n a_n(t) \, e^{-\frac{i}{\hbar} E_n t} \, | \psi_n \rangle$$

⇑

$a_n(t)$ VARIES NOW WITH TIME DUE TO $\hat{H}_1$

$$i\hbar \frac{\partial}{\partial t} \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t}\, |\psi_n\rangle$$

$$= \left(\hat{H}_0 + \hat{H}_1\right) \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t}\, |\psi_n\rangle$$

$$\Downarrow$$

$$\cancel{\sum_n a_n(t)\, E_n\, e^{-\frac{i}{\hbar}E_n t}}\, |\psi_n\rangle + \sum_n \left(i\hbar \frac{\partial a_n}{\partial t}\right) e^{-\frac{i}{\hbar}E_n t}\, |\psi_n\rangle$$

$$= \cancel{\sum_n a_n(t)\, E_n\, e^{-\frac{i}{\hbar}E_n t}}\, |\psi_n\rangle + \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t}\, \hat{H}_1\, |\psi_n\rangle$$

$$\Downarrow \qquad \langle\psi_m|$$

$$i\hbar \left(\frac{d}{dt} a_m(t)\right) e^{-\frac{i}{\hbar}E_m t} = \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t} \langle\psi_m|\hat{H}_1|\psi_n\rangle$$

$$\boxed{i\hbar \frac{da_m}{dt} = \sum_n a_n(t) \langle\psi_m|\hat{H}_1|\psi_n\rangle\, e^{+\frac{i}{\hbar}(E_m - E_n)t}}$$

↳ ASSUME AT $t \leq 0$ SYSTEM IS IN STATE $|\psi_n\rangle$

$a_n(t \leq 0) = 1$, $a_m(t \leq 0) = 0$ FOR $m \neq n$

FOR $\hat{H}_1 \ll \hat{H}_0$ APPLY 1° ORDER PERTURBATION THEORY

i.e. ON RHS APPROXIMATE $a_n(t > 0) \approx a_n(t = 0) = 1$.

$a_m(t > 0) \approx a_m(t = 0) = 0,\ m \neq n$

$$\therefore \quad i\hbar \frac{d}{dt} a_m \approx \langle \psi_m | \hat{H}_1 | \psi_n \rangle e^{\frac{i}{\hbar}(E_m - E_n)t}$$

$$\boxed{a_m(t) \approx -\frac{i}{\hbar} \int_{t'=0}^{t'=t} dt' \langle \psi_m | \hat{H}_1 | \psi_n \rangle e^{\frac{i}{\hbar}(E_m - E_n)t'}}$$

$$\Downarrow$$

↳ IF $\hat{H}_1$ DOES NOT DEPEND ON TIME

$$a_m(t) \approx -\frac{i}{\hbar} \langle \psi_m | \hat{H}_1 | \psi_n \rangle \int_{t'=0}^{t'=t} dt' \, e^{\frac{i}{\hbar}(E_m - E_n)t'}$$

$$\boxed{a_m(t) \approx + \frac{1}{(E_n - E_m)} \langle \psi_m | \hat{H}_1 | \psi_n \rangle \left(e^{\frac{i}{\hbar}(E_m - E_n)t} - 1 \right)}$$

TRANSITION PROBABILITY FROM $n \to m$ IN 1° ORDER P.T.

$$\boxed{P_{n \to m}(t) = | a_m(t) |^2 = | \langle \psi_m | \hat{H}_1 | \psi_n \rangle |^2 \frac{\sin^2\left(\frac{E_m - E_n}{2\hbar} t \right)}{\left(\frac{E_n - E_m}{2} \right)^2}}$$

↳ FERMI'S GOLDEN RULE

⇝ TAKE LONG TIME LIMIT $t \to \infty$

USE $$\left\| \frac{\sin^2(xt)}{\pi t x^2} \right|_{t\to\infty} = \delta(x)$$

$$x \neq 0 \quad \Rightarrow \quad \left. \frac{\sin^2(xt)}{\pi t x^2} \right|_{t\to\infty} \longrightarrow 0$$

$$\int_{-\infty}^{+\infty} dx \; \frac{\sin^2(xt)}{\pi t x^2} = \frac{2}{\pi} \int_0^{\infty} d(tx) \; \frac{\sin^2(xt)}{(xt)^2} = 1$$

$$\rightsquigarrow \quad P_{n\to m}(t \gg) \simeq \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2 \frac{\pi t}{\hbar^2} \underbrace{\delta\left(\frac{E_m - E_n}{2\hbar}\right)}_{2\hbar\, \delta(E_m - E_n)}$$

$$\boxed{\frac{P_{n\to m}(t\gg)}{t} = \frac{2\pi}{\hbar}\, \delta(E_m - E_n) \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2}$$

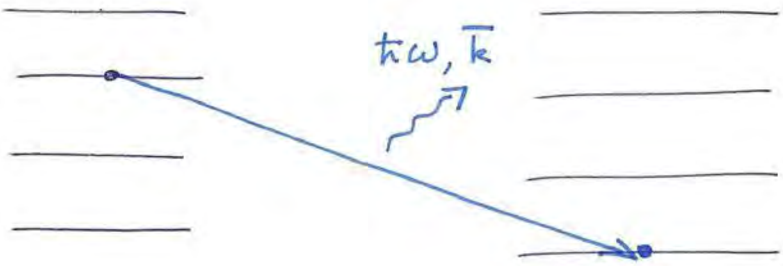
TRANSITION PROBABILITY PER UNIT OF TIME

ENERGY CONSERVATION

THIS IS KNOWN AS FERMI'S GOLDEN RULE

EMISSION OF RADIATION BY AN EXCITED ATOM

↳



$|\Psi_i\rangle$ INITIAL

$|\Psi_f\rangle$ FINAL

$| \ldots n_{\bar{k}\sigma} \ldots \rangle$

$| \ldots \underbrace{n_{\bar{k}\sigma} + 1}_{} \ldots \rangle$

ONE EXTRA PHOTON WITH $\bar{k}\,\sigma$

$$|i\rangle \equiv |\Psi_i\rangle \otimes | \ldots n_{\bar{k}\sigma} \ldots \rangle$$

⇑ MATTER STATE (e.g. e^- IN 2° EXCITED STATE OF ATOM)

$$|f\rangle \equiv |\Psi_f\rangle \otimes | \ldots n_{\bar{k}\sigma} + 1 \ldots \rangle$$

↳ TRANSITION PROBABILITY PER UNIT TIME $\frac{dP}{dt}$

$$\boxed{\frac{dP}{dt} = \frac{2\pi}{\hbar} \left| \mathcal{M}_{fi} \right|^2 \delta(E_i - E_f)}$$

(FERMI'S GOLDEN RULE)

$\hookrightarrow$ TRANSITION AMPLITUDE $\mathcal{M}_{fi}$

IN 1° ORDER PERTURBATION THEORY

$$\mathcal{M}_{fi} = \langle f \,|\, H'_{INT} \,|\, i \rangle$$

$$= \langle \Psi_f | \langle \dots n_{\bar{k}\sigma}+1 \dots | \hat{H}'_{INT} | \dots n_{\bar{k}\sigma} \dots \rangle |\Psi_i\rangle$$

$$= -\frac{e}{mc} \sum_{\bar{k}'\sigma'} N_{k'} \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}'\sigma'}$$

$$\otimes \langle \dots n_{\bar{k}\sigma}+1 \dots | \hat{a}_{\bar{k}'\sigma'} e^{i\bar{k}'\cdot\bar{x}} + \hat{a}^{+}_{\bar{k}'\sigma'} e^{-i\bar{k}'\cdot\bar{x}} | \dots n_{\bar{k}\sigma} \rangle \otimes |\Psi_i\rangle$$

$$= -\frac{e}{mc} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma}$$

$$\underbrace{\otimes \langle \dots n_{\bar{k}\sigma}+1 \dots | \hat{a}^{+}_{\bar{k}\sigma} e^{-i\bar{k}\cdot\bar{x}} | \dots n_{\bar{k}\sigma} \dots \rangle}_{e^{-i\bar{k}\cdot\bar{x}} \sqrt{n_{\bar{k}\sigma}+1}} \otimes |\Psi_i\rangle$$

$$\therefore \boxed{\mathcal{M}_{fi} = -\frac{e}{mc} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \sqrt{n_{\bar{k}\sigma}+1} \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} e^{-i\bar{k}\cdot\bar{x}} | \Psi_i \rangle}$$

$$\therefore \quad \frac{dP}{dt} = \frac{2\pi}{\hbar} \, \delta(E_i - E_f) \, |\mathcal{M}_{fi}|^2$$

$$E_i = \underset{\substack{\uparrow \\ \text{MATTER}}}{E_{Mi}}$$

$$E_f = E_{Mf} + \hbar\omega$$

$$\boxed{\begin{aligned} \frac{dP}{dt} = {} & \frac{2\pi}{\hbar} \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{(mc)^2} \cdot \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right) \cdot (n_{\bar{k}\sigma} + 1) \\ & \cdot \left| \langle \psi_f | \, \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} \, e^{-i\bar{k}\cdot\bar{x}} \, | \psi_i \rangle \right|^2 \\ & \cdot \delta(E_{Mi} - E_{Mf} - \hbar\omega_k) \end{aligned}}$$

$$\Rightarrow \quad \frac{dP}{dt} \sim (n_{\bar{k}\sigma} + 1)$$

$\therefore$ THE MORE PHOTONS AVAILABLE WITH $\bar{k}\sigma$ THE MORE LIKELY THE EMISSION OF ANOTHER ONE (BOSONS!)

↳ STIMULATED EMISSION (→ LASER)

$\therefore$ FOR $n_{\bar{k}\sigma} = 0$, <u>NON</u>-ZERO PROBABILITY

↳ SPONTANEOUS EMISSION.

- LIFE TIME OF EXCITED STATE (SPONTANEOUS EMISSION)

$$\downarrow \qquad \qquad \underline{n_{\bar{k}\sigma} = 0}$$

$$\tau$$

$$\hookrightarrow \left(\frac{1}{\tau_{i\to f}}\right) \equiv \frac{2\pi}{\hbar} \sum_{\bar{k}\sigma} \left|\langle f | H'_{INT} | i \rangle\right|^2 \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$|\bar{k}| = \frac{\omega_k}{c}$$

$$\downarrow$$

$$\frac{1}{\tau_{i\to f}} = \frac{2\pi}{\hbar} \cdot \frac{e^2}{4\pi} \cdot \frac{2\pi\hbar}{m^2 \quad L^3} \cdot$$

$$\cdot \sum_{\bar{k}\sigma} \frac{1}{\omega_k} \cdot \left|\langle \psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} \, e^{-i\bar{k}\cdot\bar{x}} | \psi_i \rangle\right|^2$$

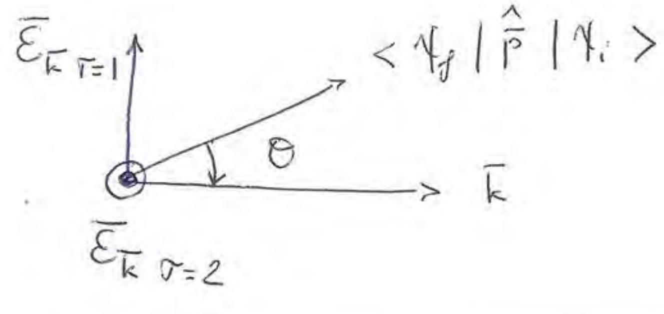
$$\cdot \, \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\downarrow \qquad \sum_{\bar{k}} \cdots = \frac{L^3}{(2\pi)^3} \int d^3\bar{k} \quad \cdots$$

$$\frac{1}{\tau_{i\to f}} = \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \, m^2} \int d^3\bar{k} \sum_{\sigma=1,2} \frac{1}{\omega_k} \, \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\cdot \left|\langle \psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} \, e^{-i\bar{k}\cdot\bar{x}} | \psi_i \rangle\right|^2$$

$\downarrow$ CHOOSE $\bar{\varepsilon}_{\bar{k}\,\sigma=1}$ IN PLANE $\bar{k}$, $\langle \psi_f | \hat{\bar{p}} | \psi_i \rangle$



$$\bar{\varepsilon}_{\bar{k}\,\sigma=2} \cdot \langle \Psi_f | \hat{\bar{p}}\, e^{-i\bar{k}.\bar{x}} | \Psi_i \rangle = 0$$

$\hookrightarrow$ FOR ATOMS $\quad \hbar \omega_k \approx 10\ eV \quad$ (TYPICALLY)

$$\bar{k}.\bar{x} \approx \frac{2\pi}{\lambda}\, a_{BOHR}$$

$$= \frac{\hbar \omega_k}{\hbar c}\, a_{BOHR}$$

$\hbar c \approx 1970\ eV\ \text{Å}$

$(1\ \text{Å} = 10^{-10}\ m)$

$$\approx \frac{10\ eV}{1970\ eV\ \text{Å}} \cdot \underbrace{a_{BOHR}}_{\approx 0.5\ \text{Å}}$$

$$\approx 2.5\ 10^{-3} \ll \underline{1}$$

LONG - WAVELENGTH APPROXIMATION

$$e^{-i\bar{k}.\bar{x}} = \underline{1} - i\bar{k}.\bar{x} + \frac{1}{2}(i\bar{k}\,\bar{x})^2 + \dots$$

$$\approx \underline{1} \qquad (\bar{k}.\bar{x} \ll 1)$$

$$\Downarrow$$

$$\bar{\varepsilon}_{\bar{k}\,\sigma=1} \cdot \langle \Psi_f | \hat{\bar{p}}\, e^{-i\bar{k}.\bar{x}} | \Psi_i \rangle$$

$$\approx \bar{\varepsilon}_{\bar{k}\,\sigma=1} \cdot \langle \Psi_f | \hat{\bar{p}} | \Psi_i \rangle$$

$$= \sin\theta\ |\langle \Psi_f | \hat{\bar{p}} | \Psi_i \rangle|$$

$$\hookrightarrow \quad \frac{1}{\tau_{i\to f}} \simeq \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \cdot m^2}$$

$$\cdot \int d^3\bar{k} \; \frac{1}{\omega_k} \; \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\cdot \sin^2\theta \; \left| \langle \Psi_f \,|\, \hat{p} \,|\, \Psi_i \rangle \right|^2$$

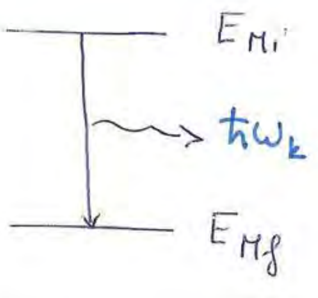
$\downarrow$ CHOOSE Z-AXIS ALONG $\langle \Psi_f | \hat{p} | \Psi_i \rangle$

$$= \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \cdot m^2} \cdot \left| \langle \Psi_f \,|\, \hat{p} \,|\, \Psi_i \rangle \right|^2$$

$$\cdot (2\pi) \underbrace{\int_{-1}^{1} d\cos\theta \; (1 - \cos^2\theta)}_{\frac{4}{3}} \cdot \int_0^\infty d|\bar{k}| \; \frac{|\bar{k}|^2}{|\bar{k}|c} \; \underbrace{\delta\left(E_{M_i} - E_{M_f} - \hbar c|\bar{k}|\right)}_{\frac{1}{\hbar c}\delta\left(|\bar{k}| - \frac{E_{M_i} - E_{M_f}}{\hbar c}\right)}$$

$$= \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{m^2} \frac{1}{\hbar c^2} \cdot \left(\frac{E_{M_i} - E_{M_f}}{\hbar c}\right) \cdot \frac{4}{3} \left| \langle \Psi_f \,|\, \hat{p} \,|\, \Psi_i \rangle \right|^2$$

$$\boxed{\frac{1}{\tau_{i\to f}} = \frac{4}{3} \cdot \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{\hbar m^2 c^3} \left(\frac{E_{M_i} - E_{M_f}}{\hbar}\right) \cdot \left| \langle \Psi_f \,|\, \hat{p} \,|\, \Psi_i \rangle \right|^2}$$



$$(E_{M_i} - E_{M_f}) \uparrow \;\Rightarrow\; \tau_{i\to f} \downarrow$$

$$\langle \psi_f | \hat{p} | \psi_i \rangle$$

$$= m \frac{d}{dt} \langle \psi_f | \hat{x} | \psi_i \rangle$$

$$= -\frac{im}{\hbar} \langle \psi_f | \left[\hat{x}, \hat{H}_M \right] | \psi_i \rangle$$

$$= m \left(-\frac{i}{\hbar} \right) \left(E_{Mi} - E_{Mf} \right) \langle \psi_f | \hat{x} | \psi_i \rangle$$

$$\therefore \boxed{\frac{1}{\tau_{i \to f}} = \frac{4}{3} \left(\frac{e^2}{4\pi} \right) \cdot \frac{1}{\hbar c^3} \left(\frac{E_{Mi} - E_{Mf}}{\hbar} \right)^3 \left| \langle \psi_f | \hat{x} | \psi_i \rangle \right|^2}$$

$e\hat{x}$: ELECTRIC DIPOLE OPERATOR

$\therefore$ ELECTRIC DIPOLE APPROXIMATION.

HYPERFINE SPLITTING IN HYDROGEN

21 cm LINE

- e^- WITH SPIN HAS $\bar{\mu}$

$$\bar{\mu} = \frac{e}{2m}\frac{\hbar}{c}\,\bar{\sigma}$$

WHICH INTERACTS WITH $\bar{B}$ AS

$$\hat{H}_{INT}''' = -\bar{\mu}\,.\,\hat{\bar{B}}$$

- EXPRESS $\hat{H}_{INT}'''$ THROUGH ITS NORMAL MODE EXPANSION.

$$\hat{H}_{INT}''' = -\frac{e\hbar}{2mc}\,\bar{\sigma}.\,(\bar{\nabla}\times\hat{\bar{A}})$$

$$\hat{\bar{A}} = \sum_{\bar{k}\sigma} N_k\,\bar{\varepsilon}_{\bar{k}\sigma}\left\{\hat{a}_{\bar{k}\sigma}\,e^{i\bar{k}.\bar{x}} + \hat{a}^{+}_{\bar{k}\sigma}\,e^{-i\bar{k}.\bar{x}}\right\}$$

$$\bar{\nabla}\times\hat{\bar{A}} = i\sum_{\bar{k}\sigma} N_k\left(\bar{k}\times\bar{\varepsilon}_{\bar{k}\sigma}\right)\left\{\hat{a}_{\bar{k}\sigma}\,e^{i\bar{k}.\bar{x}} - \hat{a}^{+}_{\bar{k}\sigma}\,e^{-i\bar{k}.\bar{x}}\right\}$$

$$\boxed{\hat{H}_{INT}''' = -\frac{ie\hbar}{2mc}\sum_{\bar{k}\sigma} N_k\,\bar{\sigma}.\,(\bar{k}\times\bar{\varepsilon}_{\bar{k}\sigma})\,.\left\{\hat{a}_{\bar{k}\sigma}\,e^{i\bar{k}.\bar{x}} - \hat{a}^{+}_{\bar{k}\sigma}\,e^{-i\bar{k}.\bar{x}}\right\}}$$

- CONSIDER e^- IN $1s$ STATE OF H

  SPIN OF e^- INTERACTS WITH SPIN OF PROTON

  (BOTH ARE SPIN 1/2)

  ⇒ RESULTING IN SPLITTING OF GROUND STATE (HYPERFINE SPLITTING)

  $e^-\uparrow$ AND $p\uparrow$ HAS SLIGHTLY <u>HIGHER</u> ENERGY

  THAN $e^-\uparrow$ AND $p\downarrow$

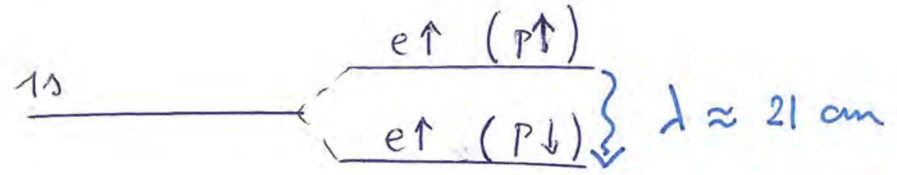
⇝ CONSIDER A TRANSITION BETWEEN

e^- IN INITIAL STATE $|\psi_i\rangle = |1s\rangle\,|\uparrow\rangle_e\,|\uparrow\rangle_p$

AND

e^- IN FINAL STATE $|\psi_f\rangle = |1s\rangle \frac{1}{\sqrt{2}}\left\{|\uparrow\rangle_e|\downarrow\rangle_p - |\downarrow\rangle_e|\uparrow\rangle_p\right\}$

⇝ IN THIS TRANSITION, A PHOTON IS EMITTED WITH $\lambda \approx 21$ cm



⇝ <u>QUESTION</u> : CALCULATE LIFETIME OF TRANSITION

<u>HELP</u> : CALCULATE MATRIX ELEMENT

$\langle f |\, H_{INT}^{'''} \,| i \rangle$ IN DIPOLE APPROX.

AND APPLY FERMI'S GOLDEN RULE

$\rightsquigarrow$ ANSWER

$$\langle f | \hat{H}'''_{INT} | i \rangle \quad \text{WITH} \quad |i\rangle = |\Psi_i\rangle \; |0\rangle_{\text{PHOTONS}}$$

$$|f\rangle = |\Psi_f\rangle \; |\bar{k}\sigma\rangle_{\text{PHOTONS}}$$

$$\hookrightarrow \frac{1}{\tau_{i\to f}} = \frac{2\pi}{\hbar} \sum_{\bar{k}\sigma} \left| \langle f | \hat{H}'''_{INT} | i \rangle \right|^2 \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\hookrightarrow \langle f | \hat{H}'''_{INT} | i \rangle = \frac{ie\hbar}{2mc} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \cdot \langle \Psi_f | \bar{\sigma} \cdot (\bar{k} \times \bar{\varepsilon}_{\bar{k}\sigma}) | \Psi_i \rangle$$

$$\left(\text{DIPOLE APPROX } e^{-i\bar{k}\cdot\bar{x}} \approx 1\right)$$

$$\hookrightarrow \frac{1}{\tau_{i\to f}} = \frac{2\pi}{\hbar} \frac{e^2\hbar^2}{4m^2c^2} \cdot \left(\frac{\hbar c^2}{2L^3}\right) \cdot \frac{L^3}{(2\pi)^3} \int d^3\bar{k} \sum_\sigma \frac{1}{(kc)} \cdot \frac{1}{\hbar c} \delta(k - k_0) \cdot \left| \langle \Psi_f | \bar{\sigma} \cdot (\bar{k} \times \bar{\varepsilon}_{\bar{k}\sigma}) | \Psi_i \rangle \right|^2$$

WITH $\left\| \; k_0 = \frac{1}{\hbar c}\left(E_{M_i} - E_{M_f}\right) = \frac{2\pi}{\underset{\underset{\underline{\underline{21\text{ cm}}}}{\parallel}}{\lambda}} \right.$

$$\hookrightarrow \mathcal{M}_{fi} = \langle \Psi_f \,|\, \bar{\sigma} \cdot (\bar{k} \times \bar{\varepsilon}_{\bar{k}\sigma}) \,|\, \Psi_i \rangle$$

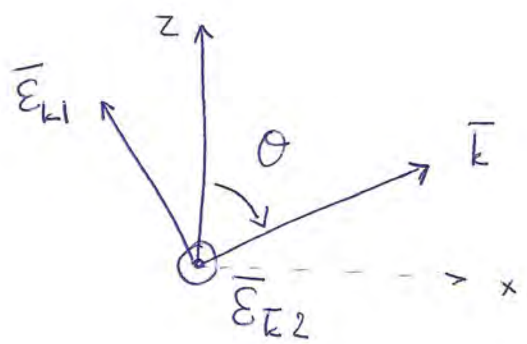
IN $|\Psi_i\rangle \Rightarrow |\uparrow\rangle_p$

IN $|\Psi_f\rangle$ ONLY TERM WITH $|\uparrow\rangle_p$ CONTRIBUTES

$$\mathcal{M}_{fi} = -\frac{1}{\sqrt{2}} \langle 1s, e\downarrow \,|\, \bar{\sigma} \cdot (\bar{k} \times \bar{\varepsilon}_{\bar{k}\sigma}) \,|\, 1s, e\uparrow \rangle$$

$\downarrow$

Z - AXIS ALONG e^- SPIN



$$(\bar{k} \times \bar{\varepsilon}_{\bar{k}\,\sigma=1}) = -|\bar{k}|\,\hat{e}_y$$

$$(\bar{k} \times \bar{\varepsilon}_{\bar{k}\,\sigma=2}) = |\bar{k}|(\cos\theta\,\hat{e}_x - \sin\theta\,\hat{e}_z)$$

$\sigma = 1$ $\qquad \mathcal{M}_{fi} = +\frac{1}{\sqrt{2}}\,|\bar{k}|\; i$

$\sigma = 2$ $\qquad \mathcal{M}_{fi} = -\frac{1}{\sqrt{2}}\,|\bar{k}|\cos\theta$

$$\hookrightarrow \frac{1}{\tau_{i\to f}} = \frac{e^2\hbar^2}{8m^2}\frac{1}{(2\pi)^2}\cdot\frac{1}{\hbar c^2}\cdot k_0^3\cdot(2\pi)$$

$$\cdot\underbrace{\int_{-1}^{1} d\cos\theta \quad \frac{1}{2}\left(1+\cos^2\theta\right)}_{\frac{4}{3}}$$

$$\boxed{\frac{1}{\tau_{i\to f}} = \left(\frac{e^2}{4\pi}\right)\cdot\frac{1}{3}\,\frac{\hbar}{m^2c^2}\,k_0^3}$$

$$= \left(\frac{e^2}{4\pi\hbar c}\right)\cdot\frac{1}{3}\,\frac{(\hbar c)^2}{(mc^2)^2}\,c\,k_0^3$$

$$\tau_{i\to f} = (137)\cdot 3\cdot\frac{(0.5\;10^6\;eV)^2}{(1.97\;10^{-7}\,eVm)^2}\cdot\frac{1}{(2\pi)^3}\frac{(21\;10^{-2}\,m)^3}{\left(3.10^8\,\frac{m}{s}\right)}$$

$$\approx 3.3\;10^{14}\;sec$$

$$\approx 10^7\;years$$

$\Longrightarrow$ RELEVANT IN RADIO ASTRONOMY ! (MICROWAVES)

CAN PENETRATE DUST CLOUDS THAT ARE OPAQUE TO VISIBLE WAVELENGTHS