Exercise sheet 7
Theoretical Physics 5 : SS 2023

29.05.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (100 points): Dirac particle in a scalar potential

Consider a Dirac particle traveling along the z-axis and subject to the scalar square-well potential

$$V(z) = \begin{cases} 0 & \text{when} \quad z < -a/2 & \text{(region I)} \\ V_0 & \text{when} \quad -a/2 \leq z \leq a/2 & \text{(region II)} \\ 0 & \text{when} \quad a/2 < z & \text{(region III)} \end{cases}$$

where $a > 0$ and $V_0 < 0$. In regions I and III, the Dirac equation takes the form

$$\left(\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}}\, c + \beta m_0 c^2\right) \psi = E\psi,$$

while in region II it takes the form

$$\left[\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}}\, c + \beta(m_0 c^2 + V_0)\right] \psi = E\psi.$$

In region II, due to the potential, the particle has an effective mass $m_{\text{eff}} = m_0 + V_0/c^2$.

a) (*30 p.*) Write down the general solution $\psi(z)$ in each of the three regions for a particle with the spin in the z-direction.

Hints: A plane-wave solution with momentum $\boldsymbol{p}$, mass m and spin label s can be written as

$$u(\vec{p}, s) = A \begin{pmatrix} \chi_s \\ \frac{c\ \boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+mc^2}\ \chi_s \end{pmatrix} e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Et)},$$

where A is some complex number and χ_s a two-component spinor. For the spin projection along the z-axis, χ_s is an eigenstate of the Pauli matrix σ_3. Do not forget that plane waves can travel in both directions. Remember that the matrices $\boldsymbol{\alpha}$ and β in standard representation are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) (*40 p.*) Impose the continuity condition at $z = \pm a/2$. Express the coefficient of the incoming plane wave of region I in terms of the plane waves of region III. Define, for convenience, the dimensionless quantity

$$\gamma := \frac{k_1 c}{E + m_0 c^2}\,\frac{E + m_{\text{eff}} c^2}{k_2 c},$$

where k_1 is the momentum in regions I and III, and k_2 is the momentum in region II.

c) (*30 p.*) Consider the special case $|m_{\text{eff}} c^2| < |E| < m_0 c^2$ corresponding to bound states. Show that these states satisfy

$$k_2 \cot\left(\frac{k_2 a}{\hbar}\right) \quad = \quad -\left(\frac{m_0 V_0}{\kappa_1} + \kappa_1\right),$$

where $\kappa_1 = -ik_1$.
Hints: Show that in the considered case there can be neither an incoming wave in region I nor an outgoing wave in region III. Then, show that continuity demands

$$\text{Im}\left(\frac{1+\gamma}{1-\gamma}\, e^{-\frac{i}{\hbar}k_2 a}\right) \quad = \quad 0,$$

and that $\gamma = i\Gamma$ is imaginary, which leads to

$$\cot\left(\frac{k_2 a}{\hbar}\right) \quad = \quad \frac{1-\Gamma^2}{2\Gamma}.$$