Exercise sheet 6
Theoretical Physics 5 : SS 2023

22.05.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (30 points): The principle of minimal substitution and the relativistic formulation of the Maxwell equations

In this exercise you will derive the relativistic (and manifestly Lorentz invariant) form of the Maxwell equations. We can use this form, and the principle of minimal substitution, to study electromagnetic interactions involving Klein-Gordon or Dirac fields (exercises 2 and 3).

a) (*10 p.*) Given the Lorentz tensor of the electromagnetic field strength,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{1}$$

the four-vector of the current, $j = (j^\mu) = (\rho, \boldsymbol{j})$, and the Levi-Civita symbol in four dimensions,

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even permutations of } (\mu,\nu,\alpha,\beta) = (0,1,2,3), \\ -1 & \text{for odd permutations of } (\mu,\nu,\alpha,\beta) = (0,1,2,3), \\ 0 & \text{else,} \end{cases} \tag{2}$$

show that the equations

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{3}$$
$$\varepsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} = 0, \tag{4}$$
$$\partial_\mu j^\mu = 0, \tag{5}$$

are equivalent to the Maxwell equations,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho, \qquad \text{Gauss' law,} \tag{6}$$
$$\frac{\partial}{\partial t}\boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{j}, \qquad \text{Ampere's law,} \tag{7}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \boldsymbol{0}, \qquad \text{no magnetic monopoles,} \tag{8}$$
$$\frac{\partial}{\partial t}\boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}, \qquad \text{Far,aday's law} \tag{9}$$

and current conservation

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0. \tag{10}$$

Hint: remember that

$$(\partial_\mu) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \quad (\partial^\mu) = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \tag{11}$$

b) *(10 p.)* We may combine the electromagnetic potentials into $(A^\mu) = (\phi, \boldsymbol{A})$. Show that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{12}$$

is equivalent to

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}, \quad \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \boldsymbol{\nabla}\phi. \tag{13}$$

c) *(10 p.)* Let q be a charge. Using the principle of minimal substitution,

$$p^\mu \to p^\mu - qA^\mu, \tag{14}$$

we can introduce interactions with the electromagnetic field.

Starting from a free, nonrelativistic particle with

$$H_{\text{free}} = \frac{\boldsymbol{p}^2}{2m} \tag{15}$$

one gets

$$H = \frac{[\boldsymbol{p} - q\boldsymbol{A}(t,\boldsymbol{x})]^2}{2m} + q\phi(t,\boldsymbol{x}). \tag{16}$$

Derive Newton's equation of motion from the Lorentz force,

$$m\ddot{\boldsymbol{x}} = q\left(\boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B}\right), \tag{17}$$

using Hamilton's equations,

$$\dot{x}^i = \frac{\partial H}{\partial p^i}, \quad \dot{p}^i = -\frac{\partial H}{\partial x^i},$$

where an overdot indicates a derivative to time.

Exercise 2. (30 points): Pionic atoms

A pionic atom is formed when a negative pion π^-, which is a spin-0 boson, is stopped in matter and is captured by an atom. The incident pion slows down by successive electromagnetic interactions with the electrons and nuclei. When the pion reaches the typical velocity of atomic electrons, the pion ejects a bound electron from its Bohr orbit and the pion is captured instead.

Let us approximate the potential between the nucleus and the pion by a square-well $V = -V_0$ for $r \leq R$ and $V = 0$ for $r > R$, where R is the nucleus radius.

a) (*10 p.*) Using the principle of minimal substitution $p_\mu \to p_\mu - \frac{e}{c}A_\mu$, with $A_\mu = (V, \mathbf{0})$, show that the Klein-Gordon equation leads to the following radial equation for the field

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2} + k^2\right] u(r) = 0, \tag{18}$$

where

$$k^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon - eV)^2 - m_\pi^2 c^4\right],$$

ϵ the energy of the pion.

Hint: How is the derivative operator defined?

Hint: Use the Klein-Gordon field in the following factorized form

$$\phi(\boldsymbol{x}, t) = u(r)\, Y_l^m(\Omega)\, e^{-\frac{i}{\hbar}\epsilon t} \tag{19}$$

and recall, from your introductory quantum mechanics classes, some properties of the spherical harmonics.

b) (*10 p.*) For a bound state we have $k^2 > 0$ for $r \leq R$ and $k^2 < 0$ for $r > R$. In both regions, solve the Eq. 18 for an s-state ($l = 0$).

Hint: Use the ansatz $u(r) = v(r)/r$.

c) (*10 p.*) Match the solutions in both regions by imposing equal logarithmic derivatives,

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\bigg|_{\text{interior solution},r=R} = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\bigg|_{\text{exterior solution},r=R},$$

and show that this matching amounts to solving the transcendental equation

$$k_i \cot(k_i R) = -k_o,$$

where $k_i^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon + eV_0)^2 - m_\pi^2 c^4\right]$ and $k_o^2 = \frac{1}{\hbar^2 c^2}\left(m_\pi^2 c^4 - \epsilon^2\right)$, with i standing for interior and o for exterior. You do not have to explicitly solve the equation.

Exercise 3. (40 points): The spectrum of relativistic electrons in a constant magnetic field

To study the behavior of relativistic electrons in a constant magnetic field, we have to solve the stationary-state Dirac equation

$$\left(-i\hbar c\, \boldsymbol{\alpha} \cdot \boldsymbol{D} + \beta mc^2\right)\psi = E\psi \tag{20}$$

where we have used the principle of minimal substitution, $\boldsymbol{\nabla} \to \boldsymbol{D} := \boldsymbol{\nabla} - \frac{ie}{\hbar c}\boldsymbol{A}$. Here, $\boldsymbol{A}$ is the magnetic vector potential, whose curl gives the magnetic field $\boldsymbol{B}$.

a) (*15 p.*) Verify that

$$(\boldsymbol{\alpha} \cdot \boldsymbol{D})^2 = 1\boldsymbol{D}^2 + \frac{e}{\hbar c}\boldsymbol{\Sigma} \cdot \boldsymbol{B}, \quad \text{where} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{21}$$

Note that in this equation 1 is a 4×4 identity matrix and the notation $\boldsymbol{V} \cdot \boldsymbol{W}$ means $\sum_i V_i W_i$, where V_i and W_i may be matrices, differential operators or the components of a vector.

b) (*25 p.*) Consider the case where $\boldsymbol{A} = (0, xB, 0)$ (B being a constant). Using the ansatz that solutions of Eq. 21 take the form

$$\psi = e^{i(p_y y + p_z z)/\hbar} u(x),$$

show that the energy eigenvalues E of a relativistic electron in constant magnetic induction $\boldsymbol{B}$ are given by

$$E^2 = m^2c^4 + p_z^2 c^2 + (2n+1)|eB|\hbar c \pm eB\hbar c, \quad n \in \mathbb{N}. \tag{22}$$

Hint: you are asked to compute E^2, not E.

Hint: remember that the eigenvalues of the differential operator corresponding to a harmonic oscillator, $-\partial_x^2 + \omega^2 x^2$ are $(2n+1)|\omega|$ with $n \in \mathbb{N}$.