Exercise sheet 5
Theoretical Physics 5 : SS 2023

15.05.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (70 points): From mechanics to field theory

A quantum field theory describes a quantum system with infinitely many degrees of freedom. To build such a theory we can start from a classical theory, quantize it (Schrödinger equation) and then introduce an arbitrary number of degrees of freedom. This is what you have done in the lectures on many-particle systems and when you derived the Klein-Gordon equation from the relativistic energy relation.

Of course, if we can quantize and then generalize a system (see the clockwise direction on the diagram below), then one expects that generalizing and then quantizing should give same result (the anticlockwise direction).

You are going to show that this intuition is correct and that both procedures give the same results. The purpose of this derivation is to give you an alternative view on what physical system the Klein-Gordon equation describes. You will see that a relativistic free particle is actually a oscillation (not an oscillator!) arising from an infinite collection of harmonic oscillators.

$$
\begin{tikzcd}
\boxed{\text{classical mechanics}} \arrow[r, "\text{quantization}"] \arrow[d, "\substack{\text{(arbitrarily) many}\\ \text{degrees of freedom}}"] & \boxed{\text{quantum mechanics}} \arrow[d, "\substack{\text{(arbitrarily) many}\\ \text{degrees of freedom}}"] \\
\boxed{\text{classical field theory}} \arrow[r, "\text{quantization}"] & \boxed{\text{quantum field theory}}
\end{tikzcd}
$$

The Lagrangian of a system of n coupled harmonic oscillators is given by

$$L = \sum_{i=1}^{n} \left(\frac{1}{2}\mu \dot{q}_i^2 - \frac{1}{2}\kappa q_i^2 - \frac{1}{2}\kappa' \left(q_{i+1} - q_i\right)^2 \right), \tag{1}$$

with positive constants μ, κ, κ'. An overdot means a derivative to time. The distance between two neighboring oscillators q_i and q_{i+1} is given by Δx. Periodic boundary conditions, $q_{n+1} = q_1$, are implied, however this should not have any bearing on what follows.

a) (*10 p.*) Show that the n equations of motion are given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} q_j = -\frac{\kappa}{\mu} q_j + \frac{\kappa'}{\mu} \left(q_{j+1} - 2q_j + q_{j-1}\right) \quad \text{for} \quad j = 1, \ldots, n. \tag{2}$$

b) (*10 p.*) Express the canonically-conjugate momentum p_j for the oscillator variable q_j in terms of $\dot{q}_j$ and show that the standard quantization conditions ($\hbar = 1$)

$$[p_j(t), q_k(t)] = -i\delta_{jk} \quad \text{for} \quad j, k = 1, \ldots, n, \tag{3}$$

lead to

$$[\dot{q}_j(t), q_k(t)] = -\frac{i}{\mu}\delta_{jk}. \tag{4}$$

c) (*10 p.*) Since the distance between neighboring oscillators is Δx, one can characterize the oscillators equally well with $q(x,t)$ instead of $q_j(t)$. Accordingly, $q_{j\pm 1}(t)$ should be replaced by $q(x \pm \Delta x, t)$. Show that quantization condition (4) can be expressed as

$$[\dot{q}(x,t), q(x',t)] = -\frac{i}{\mu} \int_{-\Delta x/2}^{\Delta x/2} \mathrm{d}y \, \delta(y - x' + x). \tag{5}$$

Hint: distinguish the two cases $x = x'$ and $|x - x'| = j\Delta x$ with $j = 1, 2, \ldots$

d) (*5 p.*) Introduce new constants A, m and c via

$$\mu = A\Delta x, \quad \kappa = m^2 A\Delta x = m^2\mu, \quad \kappa' = \frac{Ac^2}{\Delta x} = \frac{\mu c^2}{\Delta x^2}, \tag{6}$$

and show that the equations of motion (2) can be rewritten to

$$\frac{\partial^2}{\partial t^2} q(x,t) = -m^2 q(x,t) + \frac{c^2}{\Delta x^2}\left[q(x+\Delta x,t) - 2q(x,t) + q(x-\Delta x,t)\right]. \tag{7}$$

e) (*15 p.*) Now take the continuum limit, $n \to \infty$ and $\Delta x \to 0$ while keeping A, c and m fixed. Show that Eq. 1 becomes

$$L = \int \mathrm{d}x \left[\frac{1}{2}A\dot{q}^2 - \frac{1}{2}m^2 A q^2 - \frac{1}{2}c^2 A\left(\frac{\partial q}{\partial x}\right)^2\right], \tag{8}$$

and Eq. 7 becomes

$$\frac{\partial^2}{\partial t^2} q(x,t) = -m^2 q(x,t) + c^2 \frac{\partial^2 q(x,t)}{\partial x^2}, \tag{9}$$

and finally Eq. 5 becomes

$$[\dot{q}(x,t), q(x',t)] = -\frac{i}{A}\delta(x-x'). \tag{10}$$

Hint: for a continuous function f there exists an x_m with $a \leq x_m \leq b$ such that

$$\int_a^b \mathrm{d}x\, f(x) = f(x_m)(b-a). \tag{11}$$

This property can also be used for the delta function.

f) (*20 p.*) Finally, introduce the field $\phi = \sqrt{A}q$. Express the Lagrangian, the equations of motion and the quantization condition in terms of ϕ and convince yourself that A has disappeared. In other words, the only free parameters of this ‘free field theory’ are m and c.

Find a solution to Eq. 9 by using a plane-wave ansatz and show that m is the ground frequency of the oscillation (i.e. the smallest possible frequency). In a quantized field theory for elementary particles this ground frequency translates to the smallest possible energy: the rest mass of a particle.

Define a Lagrange density $\mathcal{L}$ such that $L = \int \mathrm{d}x\, \mathcal{L}$ and show that this is the Lagrange density of the Klein-Gordon field (do they give the same equations of motion?).

Remark: Note that it is the oscillations, and not the oscillators which – after quantization – become the particles in this field theory (for example, an oscillator q_j has a mass μ which has nothing to do with the particles).

Exercise 2. (30 points): Spacetime symmetries

In this exercise you will explore how symmetries of a Lagrangian lead to conserved currents. First, a small recap from the lecture notes. Suppose we have a Lagrangian $\mathcal{L}$. Let's look at a transformation where $\delta\mathcal{L}(x)$ is not necessarily equal to zero, but possibly equal to a total divergence, $\mathcal{L}(x) = \partial_\mu K^\mu(x)$ for some $K^\mu(x)$. By definition, the Noether current is given by

$$j^\mu = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\delta\phi(x) - K^\mu(x). \tag{12}$$

So, if we have a spacetime translation,

$$\phi(x) \to \phi(x+a) \xrightarrow{\text{infinitesimal}} \phi(x) \to \phi(x) + a^\nu\partial_\nu\phi(x) \implies \delta\phi(x) = a^\nu\partial_\nu\phi(x). \tag{13}$$

Clearly,

$$\mathcal{L}(x) \to \mathcal{L}(x+a) \implies \delta\mathcal{L}(x) = a^\nu\partial_\nu\mathcal{L}(x) = \partial_\nu\left(a^\nu\mathcal{L}(x)\right), \tag{14}$$

and thus $K^\nu(x) = a^\nu\mathcal{L}(x)$.

So, the Noether current is given by

$$\begin{aligned} j^\mu(x) &= \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\left[a^\nu\partial_\nu\phi(x) - a^\mu\mathcal{L}(x)\right] \\ &= a_\nu T^{\mu\nu}, \end{aligned} \tag{15}$$

where we have defined the stress-energy tensor,

$$T^{\mu\nu} = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi)}\partial^\nu\phi(x) - g^{\mu\nu}\mathcal{L}(x). \tag{16}$$

Of course, besides spacetime translation invariance, a quantum field theory should also be invariant under a Lorentz transformation (rotations and boosts). This is yet another symmetry which, according to Noether's theorem, will give rise to conserved currents. It is up to you to derive these currents.

a) (*10 p.*) You may take for granted that the infinitesimal form of a Lorentz transformation is given by

$$\phi(x) \to \phi(x^\mu + \delta\omega^{\mu\nu}x_\nu), \tag{17}$$

where $\delta\omega_{\mu\nu}$ are constants and the tensor $\delta\omega$ is antisymmetric in its indices. For example, a rotation about a unit vector $\hat{\boldsymbol{n}}$ with an angle θ gives $\delta\omega_{ij} = -\varepsilon_{ijk}\hat{n}_k\delta\theta$, while a boost in the same direction with rapidity η gives $\delta\omega_{i0} = \hat{n}_i\delta\eta$.

Like we've done in the example, derive expressions for $\delta\phi(x)$, $\delta\mathcal{L}(x)$. Show that

$$K^\mu = \delta\omega^{\mu\nu} x_\nu \mathcal{L}. \tag{18}$$

In deriving this expression, it might appear as if you have to pull a derivative through x^μ. If you do this, explain why it is allowed.

b) (*10 p.*) Show that the Noether current can be written as

$$j^\mu = -\frac{1}{2}\mathcal{M}^{\mu\alpha\beta}\delta\omega_{\alpha\beta}, \tag{19}$$

where

$$\mathcal{M}^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}. \tag{20}$$

Remark: for any field which is not a scalar field, the expression above gets an extra term, $B^{\mu\nu\rho}$, which is in general quite complicated and made up from the fields and their derivatives. Moreover, the stress-energy tensor ceases to be symmetric (which, for some deep reasons, is a rather bad thing). Therefore, using this new B tensor one can define another stress-energy tensor, called the Belinfante tensor, which is symmetric in its indices. What's more, it is precisely this tensor which also shows up in the field equations of Einstein's theory of general relativity as the stress-energy tensor!

Remark: the conserved charges associated with this current are

$$M^{\nu\rho} = \int \mathrm{d}^3x\, \mathcal{M}^{0\nu\rho}(x), \tag{21}$$

and these are called the generators of the Lorentz group. Like how momentum was defined from the stress-energy tensor, we can define angular momentum as

$$J_i := \frac{1}{2}\varepsilon_{ijk}M^{jk}. \tag{22}$$

Moreover, it is possible to prove that then

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, P_j] = i\varepsilon_{ijk}P_k, \tag{23}$$

which are the (hopefully) familiar commutation relations for momentum and angular momentum.

c) (*10 p.*) Consider explicitly

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi. \tag{24}$$

In the following, you should treat ϕ and $\phi^\dagger$ as independent variables.

Let us define two new real fields via,

$$\phi = \frac{1}{\sqrt{2}}\left[\phi_1 + i\phi_2\right], \quad \phi^\dagger = \frac{1}{\sqrt{2}}\left[\phi_1 - i\phi_2\right]. \tag{25}$$

Show that

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2. \tag{26}$$

The Lagrangian with ϕ and $\phi^\dagger$ is invariant under

$$\phi \to e^{-i\beta}\phi, \quad \beta \in \mathbb{R}, \tag{27}$$

with the corresponding transformation for $\phi^\dagger$. This transformation belongs to the group U(1). Meanwhile, the Lagrangian with fields ϕ_1 and ϕ_2 is invariant under

$$\phi_1 \to \phi_1\cos\beta + \phi_2\sin\beta, \quad \phi_2 \to \phi_2\cos\beta - \phi_1\sin\beta, \tag{28}$$

which is a transformation belonging to SO(2).

Derive the Noether currents belonging to each transformation and show that they are proportional to each other.

In mathematical jargon, what you have just seen is a reflection of the fact that U(1) is isomorphic to SO(2).