Exercise sheet 3
Theoretical Physics 5 : SS 2023

01.05.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (50 points): Boson number-operator

Consider the particle number operator $n = \sum_i c_i^\dagger c_i$ for a system of bosons, where $c_i^\dagger$ and c_i are the creation and annihilation operators for a boson i. The sum runs over all bosons.

a) (*10 p.*) Calculate $\left[n, c_i^\dagger c_j^\dagger\right]$.

b) (*5 p.*) Calculate $[n, c_i c_j]$.

c) (*15 p.*) Using induction calculate $\left[n, (c_i)^k\right]$.

d) (*20 p.*) Show that n commutes with the Hamiltonian

$$H = \sum_{i,j} \langle i|H_0|j\rangle \, c_i^\dagger c_j + \frac{1}{2} \sum_{i,j,k,l} \langle i,j|V|k,l\rangle \, c_i^\dagger c_j^\dagger c_k c_l.$$

What is the physical meaning of this commutation relation?

Exercise 2. (50 points): Bose-Hubbard model

The Bose-Hubbard model gives an approximate description of the physics of interacting bosons on a lattice. It can be used to study systems such as bosonic atoms on an optical

lattice, i.e. a periodic trap formed by the interference of counter-propagating laser beams. This system resembles a crystal in the sense that the atoms are in a periodic potential.

The Hamiltonian of this model is given by (Latin indices refer to lattice sites)

$$H = -t \sum_{\langle i,j \rangle} \left(c_i^\dagger c_j + c_j^\dagger c_i \right) + \frac{U}{2} \sum_i c_i^\dagger c_i \left(c_i^\dagger c_i - 1 \right),$$

where $\langle i, j \rangle$ means that the sum is restricted over first neighbors only, and $U > 0$. t and U are parameters.

1. (*20 p.*) Provide an interpretation of each term in this Hamiltonian (the operators and the parameters).
2. (*20 p.*) Show that in this model the number of particles is conserved.
3. (*10 p.*) Qualitatively discuss the limits $t \ll U$ and $t \gg U$. What kind of phenomenon can this model reproduce?