2) LORENTZ COVARIANCE OF DIRAC EQUATION

- COVARIANT FORM

WE WANT TO SHOW THAT UNDER LORENTZ TRANSFORMATION THE DIRAC EQ. KEEPS THE SAME FORM

↳ 4- VECTOR NOTATION

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \overline{\nabla} \right)$$

↳ $$\left[c\,(-i\hbar)\,\hat{\overline{\alpha}}\cdot\overline{\nabla} + \hat{\beta}\, m_0 c^2 \right] \Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

↓ MULTIPLY ON LEFT WITH $\hat{\beta}$

$$i\hbar \left(\hat{\beta}\, \frac{1}{c}\frac{\partial}{\partial t} + \hat{\beta}\,\overline{\alpha}\cdot\overline{\nabla} \right)\Psi - m_0 c\, \Psi = 0$$

⇓

INTRODUCE $\gamma^\mu \equiv (\gamma^0, \overline{\gamma})$

$$= (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

WITH

$$\gamma^0 \equiv \hat{\beta}$$

$$\overline{\gamma} \equiv \hat{\beta}\, \hat{\overline{\alpha}}$$

γ^μ ARE 4×4 MATRICES

$$\boxed{\left(i\hbar \quad \gamma^\mu \partial_\mu - m_0 c \right) \underline{\Psi} = 0}$$

OR WITH $\hat{p}_\mu = i\hbar \, \partial_\mu$

$$\boxed{\left(\gamma^\mu \hat{p}_\mu - m_0 c \right) \underline{\Psi} = 0}$$

$\therefore$ DIRAC EQ. IN COVARIANT FORM

$\hookrightarrow \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \overline{\gamma} = \begin{pmatrix} 0 & \overline{\sigma} \\ -\overline{\sigma} & 0 \end{pmatrix}$

$$\boxed{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \; \mathbb{1}_{4\times 4}}$$

$\leadsto \mu = \nu = 0 \qquad \hat{\beta}^2 = 1$ ✓

$\leadsto \mu = 0, \; \nu = i \qquad \gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$

$$\Updownarrow$$

$$\hat{\beta}^2 \hat{\alpha}_i + \hat{\beta} \hat{\alpha}_i \hat{\beta} = 0$$

$\Updownarrow$ MULTIPLY ON RIGHT WITH $\hat{\beta}$

$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0 \quad ✓$$

$\leadsto \mu = i = \nu \qquad 2 \gamma^i \gamma^i = -2$

$$\hat{\beta} \underbrace{\hat{\alpha}_i \hat{\beta}}_{-\hat{\beta}\hat{\alpha}_i} \hat{\alpha}_i = -1$$

$$\Updownarrow$$

$$\hat{\alpha}_i^2 = 1 \quad ✓$$

$\rightsquigarrow \quad \mu = i \, , \; \nu = j \quad (i \neq j)$

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 0$$

$$\Updownarrow$$

$$\hat{\beta}\hat{\alpha}_i \, \hat{\beta}\hat{\alpha}_j + \hat{\beta}\hat{\alpha}_j \, \hat{\beta}\hat{\alpha}_i = 0$$

$$\Updownarrow$$

$$+ \, \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = 0 \qquad \checkmark$$

WE FIND BACK COMMUTATION RELATIONS !

<u>NOTE</u> $\quad \gamma^0$ IS HERMITIAN

γ^i IS ANTI-HERMITIAN $\quad (\gamma^i)^\dagger = - \gamma^i$

↳ <u>'SLASH' NOTATION</u>

IT IS CONVENTIONAL TO INTRODUCE

$$\not{A} \equiv \gamma^\mu A_\mu$$

$$\rightsquigarrow \quad \not{\partial} \equiv \gamma^\mu \partial_\mu$$

$$\rightsquigarrow \quad \hat{\not{p}} \equiv \gamma^\mu \hat{p}_\mu$$

etc

<u>DIRAC EQ.</u>

$$\boxed{\left(\hat{\not{p}} - m_0 c \right) \Psi = 0}$$

• LORENTZ COVARIANCE

WE WANT TO DEMONSTRATE THAT DIRAC EQ. REMAINS COVARIANT UNDER A LORENTZ TF.

↳ LORENTZ TF

$$\underline{\underline{x'^\mu = a^\mu{}_\nu \, x^\nu}}$$

$$x'^\mu x'_\mu = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

} REMAINS INVARIANT (IS A LORENTZ INVARIANT)

$$ds^2 \equiv x'^\mu x'_\mu = x^\mu x_\mu$$

$$\Downarrow$$

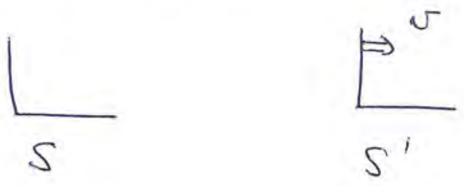
$$\underline{\underline{a^\mu{}_\nu \, a_\mu{}^\lambda = g_\nu{}^\lambda}}$$

$$\Downarrow$$

$$\left(\det(a^\mu{}_\nu) \right)^2 = 1$$

$$\det(a^\mu{}_\nu) \begin{cases} = +1 & \text{PROPER LORENTZ TF (e.g. ROTATION, LORENTZ BOOST)} \\ = -1 & \text{IMPROPER LORENTZ TF (e.g. INVOLVES SPACE INVERSION, OR TIME REVERSAL)} \end{cases}$$

e.g. ⇒ LORENTZ BOOST ALONG X-AXIS



$$\beta = \frac{v}{c}$$

$$a^{\mu}{}_{\nu} = \left(\begin{array}{cc|cc} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

⇒ SPACE INVERSION

$$a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

⇒ TIME REVERSAL

$$a^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

↳ PROOF OF LORENTZ COVARIANCE

SYSTEM S: $\left(i\hbar \gamma^\mu \partial_\mu - m_0 c \right) \Psi(x) = 0$

UNDER LORENTZ TF. $S \to S'$ $\quad x'^\mu = a^\mu{}_\nu \, x^\nu$

SYSTEM S': $\left(i\hbar \gamma'^\mu \frac{\partial}{\partial x'^\mu} - m_0 c \right) \psi'(x') = 0$

DIRAC EQ: SAME FORM

* γ'^μ IN SYSTEM S' HAVE TO SATISFY

$$\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = 2 g^{\mu\nu} \, \mathbb{1}$$

ALL 4×4 MATRICES SATISFYING ANTI-COMMUTATION RELATIONS ARE UNITARY EQUIVALENT, i.e.

$$\gamma'^\mu = U \gamma^\mu U^{-1} \quad \text{WITH } U: \text{UNITARY TF} \quad U^{-1} = U^+$$

AS UNITARY TF. DO NOT CHANGE PHYSICS

WE CAN CHOOSE

$$\gamma'^\mu = \gamma^\mu$$

* $\psi'(x')$ & $\psi(x)$ ARE RELATED BY LINEAR TRANSFORMATION $S(a)$

SINCE BOTH DIRAC EQ. & LORENTZ TF. ARE LINEAR IN COORDINATES

$$\underline{\underline{\psi'(x') = S(a)\,\psi(x)}}$$

$$\begin{cases} x' = a\,x & \text{(SHORTHAND FOR } x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \\ x = a^{-1}\,x' \end{cases}$$

$$\psi'(x') = S(a)\,\psi(x) = S(a)\,\psi(a^{-1}x')$$

OR EQUIVALENTLY

$$\psi(x) = \underline{\underline{S^{-1}(a)}}\,\psi'(x') = S^{-1}(a)\,\psi'(a\,x)$$

INVERSE TF.

$$x = (a^{-1})\,x'$$

$$\Downarrow$$

$$\psi(x) = \underline{\underline{S(a^{-1})}}\,\psi'(x') = S(a^{-1})\,\psi'(a\,x)$$

$\therefore$ IDENTIFICATION $\quad \underline{\underline{S(a^{-1}) = S^{-1}(a)}}$

$$* \quad \left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 c \right) \psi(x) = 0$$

$$\downarrow \quad \psi(x) = S^{-1}(a)\, \psi'(x')$$

$$\left[i\hbar \gamma^\mu S^{-1}(a) \frac{\partial}{\partial x^\mu} - m_0 c \; S^{-1}(a) \right] \psi'(x') = 0$$

$\downarrow$ MULTIPLY ON LEFT BY $S(a)$

$$\left[i\hbar \Big(S(a)\, \gamma^\mu S^{-1}(a) \Big) \frac{\partial}{\partial x^\mu} - m_0 c \right] \psi'(x') = 0$$

$$\downarrow \quad \frac{\partial}{\partial x^\mu} = \underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{a^\nu{}_\mu} \frac{\partial}{\partial x'^\nu} \qquad x'^\nu = a^\nu{}_\lambda \, x^\lambda$$

$$\left[i\hbar \underbrace{\Big(S(a)\, \gamma^\mu S^{-1}(a) \Big) a^\nu{}_\mu}_{} \; \partial'_\nu - m_0 c \right] \psi'(x') = 0$$

WE REQUIRE THIS
TO BE γ^ν
SO THAT DIRAC EQ. IN TRANSFORMED FRAME
HAS SAME FORM

$$\left(i\hbar \; \gamma^\nu \partial'_\nu - m_0 c \right) \psi'(x') = 0$$

$$\therefore \quad S(a)\, \gamma^{\mu}\, S^{-1}(a)\, a^{\nu}{}_{\mu} = \gamma^{\nu}$$

OR EQUIVALENTLY

$$\boxed{S^{-1}(a)\, \gamma^{\nu}\, S(a) = a^{\nu}{}_{\mu}\, \gamma^{\mu}}$$

↳ INFINITESIMAL PROPER LORENTZ TF

$$a^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \underbrace{\Delta\omega^{\nu}{}_{\mu}}_{\text{INFINITESIMAL}}$$

INFINITESIMAL (i.e. NEGLECT TERMS QUADRATIC IN $\Delta\omega$)

$$a^{\mu}{}_{\nu}\, a_{\mu}{}^{\lambda} = g_{\nu}{}^{\lambda}$$

$$\left(g^{\mu}{}_{\nu} + (\Delta\omega)^{\mu}{}_{\nu} \right)\left(g_{\mu}{}^{\lambda} + (\Delta\omega)_{\mu}{}^{\lambda} \right) = g^{\lambda}{}_{\nu}$$

$$g^{\lambda}{}_{\nu} + (\Delta\omega)^{\lambda}{}_{\nu} + (\Delta\omega)_{\nu}{}^{\lambda} \simeq g^{\lambda}{}_{\nu}$$

$$\Downarrow$$

$$(\Delta\omega)_{\mu\nu} = -(\Delta\omega)_{\nu\mu}$$

ANTI - SYMMETRIC

↳ 6 INDEPENDENT INFINITESIMAL PROPER LORENTZ TF

3 ROTATIONS

e.g. $(\Delta\omega)^{1}{}_{2} = (\Delta\varphi) = -(\Delta\omega)^{12}$

ROTATION AROUND Z-AXIS OVER ANGLE $\Delta\varphi$

3 LORENTZ BOOSTS.

e.g. $(\Delta\omega)^{0}{}_{1} = -(\Delta\omega)^{01} = -\Delta\beta$

BOOST ALONG X-AXIS WITH VELOCITY $c(\Delta\beta)$

$S(\Delta\omega)$ CAN BE CONSTRUCTED

BY EXPANDING UP TO TERMS <u>LINEAR</u> IN $\Delta\omega$

$$S \equiv \mathbb{1} - \frac{i}{4} \underbrace{\sigma_{\mu\nu}} (\Delta\omega)^{\mu\nu}$$

DEFINES 6 INDEPENDENT 4×4 MATRICES

WITH $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$

$$S^{-1}(a)\, \gamma^{\nu}\, S(a) = a^{\nu}{}_{\mu}\, \gamma^{\mu}$$

$\Downarrow$ FOR INFINITESIMAL TF.

$$\left[1 + \frac{i}{4}\sigma_{\alpha\beta}(\Delta\omega)^{\alpha\beta}\right]\gamma^{\nu}\left[1 - \frac{i}{4}\sigma_{\alpha\beta}(\Delta\omega)^{\alpha\beta}\right]$$

$$= \gamma^{\nu} + (\Delta\omega)^{\nu}{}_{\mu}\,\gamma^{\mu}$$

$$\Downarrow$$

$$\frac{i}{4}\left[\sigma_{\alpha\beta}, \gamma^{\nu}\right](\Delta\omega)^{\alpha\beta} = (\Delta\omega)^{\nu}{}_{\mu}\,\gamma^{\mu}$$

$$= (\Delta\omega)^{\alpha\beta}\left[g^{\nu}_{\alpha}\,\gamma_{\beta}\right]$$

$$= \frac{1}{2}(\Delta\omega)^{\alpha\beta}\left[g^{\nu}_{\alpha}\,\gamma_{\beta} - g^{\nu}_{\beta}\,\gamma_{\alpha}\right]$$

BECAUSE $(\Delta\omega)^{\alpha\beta}$ IS ANTI-SYMM.

$\forall\, (\Delta\omega)^{\alpha\beta}$ $\Downarrow$

$$\left[\sigma_{\alpha\beta}, \gamma^{\nu}\right] = -2i\left[g^{\nu}_{\alpha}\,\gamma_{\beta} - g^{\nu}_{\beta}\,\gamma_{\alpha}\right]$$

DIRAC EQ. IS LORENTZ COVARIANT (FOR PROPER LORENTZ TF)
IF WE FIND 4×4 MATRIX $\sigma_{\alpha\beta}$ SATISFYING

$$\left[\sigma_{\alpha\beta} , \gamma^{\nu} \right] = -2i \left[g^{\nu}_{\alpha} \gamma_{\beta} - g^{\nu}_{\beta} \gamma_{\alpha} \right]$$

WITH $\sigma_{\alpha\beta} = - \sigma_{\beta\alpha}$

<u>SOLUTION</u>

$$\sigma_{\alpha\beta} = \frac{i}{2} \left[\gamma_{\alpha} , \gamma_{\beta} \right]$$

<u>PROOF</u>

$$\left[\sigma_{\alpha\beta} , \gamma^{\nu} \right]$$

$$= \sigma_{\alpha\beta} \gamma^{\nu} - \gamma^{\nu} \sigma_{\alpha\beta}$$

$$= \frac{i}{2} \left\{ \left(\gamma_{\alpha} \gamma_{\beta} - \gamma_{\beta} \gamma_{\alpha} \right) \gamma^{\nu} - \gamma^{\nu} \left(\gamma_{\alpha} \gamma_{\beta} - \gamma_{\beta} \gamma_{\alpha} \right) \right\}$$

$$= \frac{i}{2} \left\{ \left(\gamma_{\alpha} \gamma_{\beta} - \gamma_{\beta} \gamma_{\alpha} \right) \gamma^{\nu} - \left(2 g^{\nu}_{\alpha} \gamma_{\beta} - \gamma_{\alpha} \gamma^{\nu} \gamma_{\beta} \right) \right.$$
$$\left. + \left(2 g^{\nu}_{\beta} \gamma_{\alpha} - \gamma_{\beta} \gamma^{\nu} \gamma_{\alpha} \right) \right\}$$

$$= \frac{i}{2} \left\{ \cancel{\left(\gamma_{\alpha} \gamma_{\beta} - \gamma_{\beta} \gamma_{\alpha} \right) \gamma^{\nu}} - 2 g^{\nu}_{\alpha} \gamma_{\beta} + 2 g^{\nu}_{\beta} \gamma_{\alpha} - \cancel{\gamma_{\alpha} \gamma_{\beta} \gamma^{\nu}} \right.$$
$$\left. + 2 g^{\nu}_{\beta} \gamma_{\alpha} - 2 g^{\nu}_{\alpha} \gamma_{\beta} + \cancel{\gamma_{\beta} \gamma_{\alpha} \gamma^{\nu}} \right\}$$

$$= 2i \left\{ g^{\nu}_{\beta} \gamma_{\alpha} - g^{\nu}_{\alpha} \gamma_{\beta} \right\}$$

$$\overset{!}{=} -2i \left\{ g^{\nu}_{\alpha} \gamma_{\beta} - g^{\nu}_{\beta} \gamma_{\alpha} \right\}.$$

$$\boxed{\begin{aligned} a^{\nu}{}_{\mu} &= g^{\nu}{}_{\mu} + (\Delta\omega)^{\nu}{}_{\mu} \\ S(\Delta\omega) &= \mathbb{1} + \frac{1}{8}\left[\gamma_\mu, \gamma_\nu\right](\Delta\omega)^{\mu\nu} \end{aligned}}$$

DIRAC EQ. WILL BE LORENTZ COVARIANT
IF UNDER LORENTZ TF $x' = a\,x$
THE SPINORS TRANSFORM AS $\psi'(x') = S(a)\,\psi(x)$
WITH S AS SHOWN ABOVE

↳ FINITE PROPER LORENTZ TRANSFORMATION

* WRITE INFINITESIMAL TF AS

$$(\Delta\omega)^{\mu}{}_{\nu} = (\Delta\omega)\,(I_n)^{\mu}{}_{\nu}$$

WITH $(\Delta\omega)$ INFINITESIMAL 'ANGLE' ABOUT AXIS n

e.g. FOR BOOST ALONG x- AXIS $\quad \Delta\omega = (\Delta\beta \quad)$

$$(I_1)^{\mu}{}_{\nu} = \left(\begin{array}{cc|c} 0 & -1 & \\ -1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)$$

$$(I_1)^{0}{}_{1} = (I_1)^{1}{}_{0} = -1$$

$$(I_1)^2 = \left(\begin{array}{cc|c} 1 & & \\ & 1 & 0 \\ \hline & 0 & 0 \end{array}\right) \qquad \begin{aligned} (I_1)^3 &= I_1 \\ (I_1)^4 &= I_1^2 \\ &\vdots \end{aligned}$$

* FINITE PROPER LORENTZ TF

CAN BE WRITTEN AS SUCCESSION OF INFINITESIMAL ONES

$$x'^{\mu} = \lim_{N\to\infty} \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\mu}{}_{\alpha_1} \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\alpha_1}{}_{\alpha_2} \cdots$$

$$\cdots \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\alpha_{N-1}}{}_{\alpha_N} x^{\alpha_N}$$

NOTE $\Delta\omega = \frac{\omega}{N}$ WITH N LARGE

$$= \lim_{N\to\infty} \left[\left(\mathbb{1} + \frac{\omega}{N} I_n\right)^N\right]^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \left(e^{\omega I_n}\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \left(\underbrace{\left[1 + \frac{(\omega I_n)^2}{2!} + \frac{(\omega I_n)^4}{4!} + \dots\right]}_{\text{EVEN POWERS}} + \underbrace{\left[\omega I_n + \frac{(\omega I_n)^3}{3!} + \dots\right]}_{\text{ODD POWERS}} \right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \left(\cosh(\omega I_n) + \sinh(\omega I_n)\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$\downarrow$ BECAUSE $I_n^4 = I_n^2$

$I_n^3 = I_n$

$$= \left(\mathbb{1} - I^2 + (\cosh\omega)\, I^2 + (\sinh\omega)\, I\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

e.g. BOOST ALONG X-AXIS

$$\mathbb{1} - (I_1)^2 + (\cosh\omega)(I_1)^2 + (\sinh\omega)\, I_1$$

$$= \left[\begin{array}{cc|cc} \cosh\omega & -\sinh\omega & & \\ -\sinh\omega & \cosh\omega & & 0 \\ \hline & & 1 & \\ & 0 & & 1 \end{array}\right]$$

IDENTIFY

$$\sinh\omega = \beta\gamma$$
$$\cosh\omega = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

* FINITE SPINOR TRANSFORMATION

$$\psi'(x') = S(a)\,\psi(x)$$

$$= \lim_{N\to\infty}\left(1 - \frac{i\omega}{4N}\sigma_{\alpha\beta}\,(I_n)^{\alpha\beta}\right)^N \psi(x)$$

$$\boxed{\psi'(x') = \exp\left\{-\frac{i}{4}\,\omega\,\sigma_{\alpha\beta}\,(I_n)^{\alpha\beta}\right\}\psi(x)}$$

$\rightsquigarrow$ e.g. LORENTZ BOOST ALONG X-AXIS.

$$(I_1)^{01} = 1$$
$$(I_1)^{10} = -1$$

$$\psi'(x') = \exp\left\{-\frac{i}{2}\,\omega\,\sigma_{01}\right\}\psi(x)$$

WITH
$$\sigma_{01} = \frac{i}{2}\left(\gamma_0\gamma_1 - \gamma_1\gamma_0\right) = -i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = -i\,\alpha_1$$

$\rightsquigarrow$ ROTATION AROUND Z-AXIS. AROUND ANGLE φ

$$(I)^1{}_2 = 1 \qquad (I)^2{}_1 = -1$$
$$(I)^{12} = -1 \qquad (I)^{21} = +1$$

$$\psi'(x') = \exp\left\{+\frac{i}{2}\varphi\,\sigma_{12}\right\}\psi(x)$$

WITH
$$\sigma_{12} = \frac{i}{2}\left(\gamma_1\gamma_2 - \gamma_2\gamma_1\right) = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \Sigma_3$$

ANALOGY WITH 2-COMPONENT SPINORS φ (PAULI)

$$\varphi'(x') = \exp\left\{\frac{i}{2}\,\varphi\,\sigma_3\right\}\varphi(x)$$

$\Rightarrow$ ROTATION AROUND z-AXIS OVER ANGLE φ

(ACTIVE ROTATION)

- INFINITESIMAL ROTATION

$$x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}$$

$$a^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + (\Delta\varphi)(J_z)^{\mu}{}_{\nu}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Delta\varphi & 0 \\ 0 & -\Delta\varphi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$(J_z)^{1}{}_{2} = +1 \qquad (J_z)^{2}{}_{1} = -1$$

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(J_z)^2 = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(J_z)^3 = -(J_z)$$

$$(J_z)^4 = -(J_z)^2$$

- FINITE ROTATION (OVER ANGLE φ)

AS SUCCESSION ON N INFINITESIMAL ONES ($N \to \infty$)

$$\Delta\varphi = \frac{\varphi}{N}$$

$$x'^{\mu} = \lim_{N\to\infty} \left[\left(1 + \frac{\varphi}{N} J_z\right)^N\right]^{\mu}{}_{\nu}\, x^{\nu}$$

$$= \left(e^{\varphi J_z}\right)^{\mu}{}_{\nu}\, x^{\nu}$$

$$= \left(\left[1 + \frac{1}{2!}(\varphi J_z)^2 + \frac{1}{4!}(\varphi J_z)^4 + \dots\right] + \left[\varphi J_z + \frac{1}{3!}(\varphi J_z)^3 + \dots\right]\right)^{\mu}{}_{\nu}\, x^{\nu}$$

$$= \left(\left[1 + \frac{1}{2!}\varphi^2 J_z^2 + \frac{1}{4!}\varphi^4(-J_z^2) + \dots\right] + \left[\varphi + \frac{(-1)}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \dots\right] J_z\right)^{\mu}{}_{\nu}\, x^{\nu}$$

$$= \left(1 + J_z^2 + \left[1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 + \dots\right](-J_z^2) + \left[\varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \dots\right] J_z\right)^{\mu}{}_{\nu}\, x^{\nu}$$

$$\boxed{x'^{\mu} = \left(1 + J_z^2 + (\cos\varphi)(-J_z^2) + (\sin\varphi)\, J_z\right)^{\mu}{}_{\nu}\, x^{\nu}}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

<u>OK</u> ACTIVE ROTATION

- FINITE SPINOR ROTATION (OVER ANGLE φ)

$$\boxed{\psi'(x') = \exp\left\{-\frac{i}{4}\varphi\,\sigma_{\alpha\beta}\,(J_z)^{\alpha\beta}\right\}\psi(x)}$$

$$(J_z)^{12} = -1 \qquad (J_z)^{21} = +1.$$

$$\psi'(x') = \exp\left\{+\frac{i}{2}\varphi\,\sigma_{12}\right\}\psi(x)$$

$$\sigma_{12} = \frac{i}{2}(\gamma_1\gamma_2 - \gamma_2\gamma_1) = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \equiv \Sigma_3$$

$$\boxed{\psi'(x') = \exp\left\{\frac{i}{2}\varphi\,\Sigma_3\right\}\psi(x)}$$

NOTE CONSIDER SPINOR WITH SPIN PROJ $+\frac{\hbar}{2}$ ALONG Z-AXIS

i.e. $\Sigma_3 \psi = + \psi$ EIGENSTATE OF Σ_3

UNDER ROTATION, ψ TRANSFORMS AS

$$\psi'(x') = \exp\left(\frac{i}{2}\varphi\right) \psi(x)$$

$\therefore$ IT TAKES A ROTATION OVER $\underline{\underline{4\pi}}$ (!) BEFORE SPINOR TURNS INTO ITSELF

(DOUBLE VALUEDNESS OF SPINOR LAW OF ROTATION)

$\Downarrow$

PHYSICAL QUANTITIES MUST BE <u>BILINEARS</u> IN ψ

↳ TURN INTO THEMSELVES AFTER ROTATION OVER 2π

NOTE : FOR ROTATIONS $\sigma_{ij}^{+} = \sigma_{ij}$

$$\Rightarrow \quad S_R^{+} = S_R^{-1} \quad \text{UNITARY}$$

FOR LORENTZ BOOSTS.

$$S_B^{+} = \exp\left\{-\frac{\omega}{2}\alpha_1\right\} = S_B$$

$$S_B^{-1} = \gamma_0 S_B^{+} \gamma_0$$

(SHOW THIS YOURSELF !)

$\therefore$ FOR PROPER LORENTZ TF $\underline{\underline{S^{-1} = \gamma_0 S^{+} \gamma_0}}$

• COVARIANCE OF CONTINUITY EQUATION

↳ CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} . \bar{J} = 0$$

$$\rho = \Psi^{+} \psi \quad , \quad \bar{J} = c \, \psi^{+} \bar{\alpha} \psi$$

$$\rho = \psi^{+} \gamma^{0} \gamma^{0} \psi \quad , \quad J^{i} = c \, \psi^{+} \gamma^{0} \gamma^{i} \psi$$

NOTATION "ψ BAR" $\quad \bar{\psi} \equiv \psi^{+} \gamma^{0}$

$$J^{\mu} \equiv (c \rho , \bar{J})$$

$$\boxed{J^{\mu}(x) = c \bar{\psi}(x) \, \gamma^{\mu} \psi(x)}$$

CONT. EQUATION : $\quad \frac{\partial}{\partial x^{\mu}} J^{\mu} = 0$

↳ COVARIANCE

$$J'^{\mu}(x') = c \bar{\psi}'(x') \, \gamma^{\mu} \psi'(x')$$

$$= c \, \psi^{+}(x) \, S^{+} \gamma^{0} \gamma^{\mu} S \, \psi(x)$$

$$\Big\downarrow \text{ USING } \quad S^{-1} = \gamma_{0} S^{+} \gamma_{0}$$

$$= c \underbrace{\psi^{+}(x) \, \gamma_{0}}_{\bar{\psi}} \underbrace{S^{-1} \gamma^{\mu} S}_{a^{\mu}{}_{\nu} \gamma^{\nu}} \psi(x)$$

$$= a^{\mu}{}_{\nu} \, c \bar{\psi}(x) \, \gamma^{\nu} \psi(x) = a^{\mu}{}_{\nu} \, J^{\nu}(x)$$

$\therefore$ J^{μ} TRANSFORMS AS A FOUR-VECTOR.

- SPATIAL INVERSION

↳ IMPROPER LORENTZ TF

$$\bar{x}' = -\bar{x}$$
$$t' = t$$

$$a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

↳ DENOTE SPINOR TF S BY P:

i.e. $\psi'(x') = P\,\psi(x)$

$$P^{-1}\gamma^{\nu}P = a^{\nu}{}_{\mu}\,\gamma^{\mu}$$

$\nu = 0$ $\quad P^{-1}\gamma^{0}P = \gamma^{0}$

$\nu = i$ $\quad P^{-1}\gamma^{i}P = -\gamma^{i}$

SATISFIED IF $\underline{\underline{P = \gamma^{0}\underbrace{e^{i\varphi}}_{\text{PHASE FACTOR OF NO PHYSICAL IMPORTANCE}}}}$

$$\therefore \quad \psi'(t, -\bar{x}) = e^{i\varphi}\,\gamma^{0}\,\psi(t, \bar{x})$$

• BILINEAR COVARIANTS

↳ PHYSICAL QUANTITIES CORRESPOND WITH BILINEAR EXPRESSIONS OF SPINORS WITH 4 x 4 MATRIX INBETWEEN

i.e. $\bar{\psi} \, \Gamma \, \psi$

THERE ARE 16 INDEPENDENT BILINEAR COVARIANTS

SCALAR (1) $\quad \Gamma_S = \mathbb{1}$

VECTOR (4) $\quad \Gamma_V^{\mu} = \gamma^{\mu}$

AXIAL VECTOR (4) $\quad \Gamma_A^{\mu} = \gamma^{\mu}\gamma_5$

PSEUDO-SCALAR (1) $\quad \Gamma_P = i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

TENSOR (6) $\quad \Gamma_T^{\mu\nu} = \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$

↳ PROPERTIES

i) $\forall \Gamma : \quad \Gamma^2 = \pm \mathbb{1}$

$\rightsquigarrow \Gamma_S^2 = \mathbb{1}$

$\rightsquigarrow \Gamma_P^2 = \mathbb{1} \qquad (\gamma_5)^2 = \mathbb{1}$

$$\left.\begin{array}{l} \rightsquigarrow (\gamma^0)^2 = \mathbb{1} \\ (\gamma^i)^2 = -\mathbb{1} \end{array}\right\} \quad (\gamma^{\mu})^2 = \mathbb{1}\, g^{\mu\mu}$$

(NOT SUMMED OVER μ HERE!)

$\rightsquigarrow \gamma^{\mu}\gamma_5 + \gamma_5\gamma^{\mu} = 0$

$$(\gamma^\mu \gamma_5)^2 = \gamma^\mu \gamma_5 \; \gamma^\mu \gamma_5 \qquad \text{(NOT SUMMED OVER)}$$

$$= - \gamma^\mu \gamma_5 \gamma_5 \gamma^\mu \qquad \gamma_5^2 = \mathbb{1}$$

$$= -(\gamma^\mu)^2$$

$$= - g^{\mu\mu} \, \mathbb{1}$$

$$\rightsquigarrow \quad (\Gamma_T^{\mu\nu})^2 = -\frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

$$= -\frac{1}{4}\Big\{ \gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu \Big\}$$

$$= -\frac{1}{4}\Big\{ 2g^{\nu\mu}\gamma^\mu\gamma^\nu - (\gamma^\mu)^2(\gamma^\nu)^2 - g^{\mu\mu}(\gamma^\nu)^2 - g^{\nu\nu}(\gamma^\mu)^2 + 2g^{\nu\mu}\gamma^\nu\gamma^\mu - (\gamma^\nu)^2(\gamma^\mu)^2 \Big\}$$

$$= -\frac{1}{4}\Big\{ 4g^{\mu\nu} g^{\mu\nu} - 4 g^{\mu\mu} g^{\nu\nu} \Big\}$$

$$= \left[g^{\mu\mu} g^{\nu\nu} - (g^{\mu\nu})^2 \right] \mathbb{1}$$

$$= \begin{cases} 0 & \mu = \nu \\ g^{\mu\mu} g^{\nu\nu} \, \mathbb{1} & \mu \neq \nu \end{cases}$$

ii) $\forall \Gamma_i$ EXCEPT Γ_S :

$\exists \Gamma_j$ SUCH THAT $\Gamma_i \Gamma_j = - \Gamma_j \Gamma_i$

$$\Downarrow$$

$$\underline{\underline{\mathrm{Tr}\, \Gamma_i = 0}}$$

PROOF : $\rightsquigarrow$ $\mathrm{Tr}\, \gamma_5 = 0$

$\rightsquigarrow$ $\mathrm{Tr}\, \gamma^\mu = 0$

$\rightsquigarrow$ $\mathrm{Tr}\, \gamma^\mu \gamma_5 = 0$

$$\gamma^i \gamma_5 = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\gamma^0 \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$\rightsquigarrow$ $\mathrm{Tr}\, \sigma^{\mu\nu} = 0$

$$\mathrm{Tr} \left\{ \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right\}$$

$$= \underbrace{\mathrm{Tr} \left\{ \gamma^\mu \gamma^\nu \right\}}_{= \mathrm{Tr} \left\{ \gamma^\nu \gamma^\mu \right\}} - \mathrm{Tr} \left\{ \gamma^\nu \gamma^\mu \right\}$$

$$= 0$$

iii) $\forall \Gamma_i, \Gamma_j$ $(i \neq j)$

$\exists \Gamma_k$ $(\neq \Gamma_S)$ SUCH THAT

$$\Gamma_i \Gamma_j = c\, \Gamma_k$$ C : COMPLEX NUMBER

CHECK THIS BY DIRECT INSPECTION

iv) Γ_i ARE LINEARLY INDEPENDENT

i.e. IF $\sum_i a_i \Gamma_i = 0$

$$\Updownarrow$$

$$a_i = 0$$

$\rightsquigarrow$ $\sum_i a_i \Gamma_i = 0$

$\downarrow$ MULTIPLY BY Γ_j $(j \neq S)$

$$\sum_i a_i \Gamma_j \Gamma_i = 0$$

$\downarrow$ TAKE Tr

$$0 = \sum_i a_i \, \mathrm{Tr}\{\Gamma_j \Gamma_i\} = a_j \, \mathrm{Tr}\{\Gamma_j^2\} + \sum_{i \neq j} a_i \underbrace{\mathrm{Tr}\{\Gamma_j \Gamma_i\}}_{= 0}$$

$$= \pm 4 a_j$$

$$\Updownarrow$$

$$\underline{a_j = 0}$$

$\rightsquigarrow$ FOR $\Gamma_j = \Gamma_S$

$$\sum_i a_i \Gamma_S \Gamma_i = 0$$

$$\Downarrow$$

$$\sum_i a_i \, \mathrm{Tr}\{\Gamma_S \Gamma_i\} = 0$$

$$a_S \underbrace{\mathrm{Tr}\{\Gamma_S^2\}}_{4} + \sum_{i \neq S} a_i \underbrace{\mathrm{Tr}\{\Gamma_i\}}_{0} = 0$$

$$\Updownarrow$$

$$\underline{a_S = 0}$$

LORENTZ TRANSFORMATION OF BILINEAR COVARIANTS

- S (SCALAR)

$$\bar{\psi}'(x')\,\psi'(x') = \bar{\psi}(x)\,\psi(x)$$

PROOF

$$\psi'(x') = S\,\psi(x)$$
$$\bar{\psi}'(x') = \bar{\psi}(x)\,S^{-1}$$

- P (PSEUDO SCALAR)

$$\bar{\psi}'(x')\,\gamma_5\,\psi'(x') = (\det a)\;\bar{\psi}(x)\,\gamma_5\,\psi(x)$$

PROOF

$$\bar{\psi}'(x')\,\gamma_5\,\psi'(x')$$
$$= \bar{\psi}(x)\,S^{-1}\gamma_5\,S\,\psi(x)$$

PROPER: $S = \exp\left\{-\frac{i}{4}\,\omega\,\sigma_{\alpha\beta}\,(I_n)^{\alpha\beta}\right\}$

$$[\gamma_5, \sigma_{\alpha\beta}] = 0$$
$$\Downarrow$$
$$\gamma_5\,S = S\,\gamma_5$$
$$S^{-1}\gamma_5\,S = \gamma_5$$

IMPROPER $S = P = \gamma^0$

$$S^{-1}\gamma_5\,S = \gamma^0\gamma_5\gamma^0 = -\gamma_5(\gamma^0)^2 = -\gamma_5$$

- V (VECTOR)

$$\underline{\underline{\bar{\psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\ \bar{\psi}(x)\gamma^\nu\psi(x)}}$$

PROOF $\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)\underbrace{S^{-1}\gamma^\mu S}_{a^\mu{}_\nu\gamma^\nu}\psi(x)$

$$= a^\mu{}_\nu\ \bar{\psi}(x)\gamma^\nu\psi(x)$$

- A (AXIAL - VECTOR)

$$\underline{\underline{\bar{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = (\det a)\ a^\mu{}_\nu\ \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)}}$$

PROOF $\bar{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = \bar{\psi}(x)\ S^{-1}\gamma^\mu\underbrace{\gamma_5 S}_{S\gamma_5(\det a)}\psi(x)$

$$= a^\mu{}_\nu(\det a)\ \bar{\psi}(x)\gamma^\nu\gamma_5\psi(x)$$

- T (TENSOR)

$$\underline{\underline{\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\kappa\ a^\nu{}_\lambda\ \bar{\psi}(x)\sigma^{\kappa\lambda}\psi(x)}}$$

PROOF $\bar{\psi}'(x')\ \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi'(x')$

$$= \bar{\psi}(x)\ \frac{i}{2}S^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)S\psi(x)$$

$$= \bar{\psi}(x)\ \frac{i}{2}\Big(\underbrace{S^{-1}\gamma^\mu S}_{a^\mu{}_\kappa\gamma^\kappa}\underbrace{S^{-1}\gamma^\nu S}_{a^\nu{}_\lambda\gamma^\lambda} - S^{-1}\gamma^\nu S\,S^{-1}\gamma^\mu S\Big)\psi(x)$$

$$= a^\mu{}_\kappa\, a^\nu{}_\lambda\ \bar{\psi}(x)\ \frac{i}{2}(\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa)\psi(x)$$

- PLANE - WAVE SOLUTIONS OF DIRAC EQUATION RECONSTRUCTED FROM LORENTZ TRANSFORMATION

↳ FREE DIRAC PARTICLE AT REST

$$p^{\mu}_{(0)} \left(\frac{E}{c}, \bar{p} \right) = \left(m_0 c, \bar{0} \right)$$

↑ IN REST FRAME

$$\psi(x) = w\left(\bar{0}, s_z^{(0)}\right) e^{-\frac{i}{\hbar}\varepsilon(m_0 c^2)t}$$

- FOR PARTICLE SOLUTION $\varepsilon = +1$, $w\left(\bar{0}, s_z^{(0)}\right) = u\left(\bar{0}, s_z^{(0)}\right)$

→ IN REST FRAME

$s_z^{(0)}$: SPIN - PROJ ALONG Z - AXIS.

$$u\left(\bar{0}, s_z^{(0)} = +\frac{\hbar}{2}\right) = N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u\left(\bar{0}, s_z^{(0)} = -\frac{\hbar}{2}\right) = N \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- FOR ANTI-PARTICLE SOLUTION $\varepsilon = -1$, $w\left(\bar{0}, s_z^{(0)}\right) = v\left(\bar{0}, s_z^{(0)}\right)$

$$v\left(\bar{0}, s_z^{(0)} = -\frac{\hbar}{2}\right) = N \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v\left(\bar{0}, s_z^{(0)} = +\frac{\hbar}{2}\right) = N \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ $w\left(\bar{0}, s_z^{(0)}\right)$ EIGENFUNCTIONS OF Z - COMP. OF $\bar{\Sigma}$

↳ TO DESCRIBE PARTICLE OF FINITE MOMENTUM

$$p^\mu = \left(\frac{E}{c}, \bar{p}\right) \qquad E = \sqrt{c^2\bar{p}^2 + m_0^2 c^4}$$

PERFORM LORENTZ BOOST $\quad p^\mu = a^\mu{}_\nu \, P^\nu_{(0)}$

- POLARIZATION "4- VECTOR" IN REST FRAME

$$S^\mu_{(0)} = (0, \bar{e}_z)$$

DESCRIBES QUANTIZATION AXIS IN REST FRAME.
↳ $s_z^{(0)} = \pm \frac{\hbar}{2}$ SPIN PROJ ALONG THAT AXIS

$$S^\mu_{(0)} \cdot S_{(0)\mu} = -1$$

$$S^\mu_{(0)} \cdot P_{(0)\mu} = 0$$

- POLARIZATION 4- VECTOR IN ARBITRARY FRAME

$$S^\mu = a^\mu{}_\nu \, S^\nu_{(0)} \qquad S^\mu = (S^0, \bar{S})$$

LORENTZ INV. $\Rightarrow$ $\begin{cases} S^\mu S_\mu = -1 \\ S^\mu P_\mu = 0 \end{cases}$ $\qquad \begin{aligned} (S^0)^2 - \bar{S}^2 &= -1 \\ S^0 \frac{E}{c} - \bar{S}.\bar{P} &= 0 \end{aligned}$

$$S^\mu \left(\frac{c\,\bar{S}.\bar{P}}{E}, \bar{S}\right) \qquad \text{WITH} \quad -\frac{c^2}{E^2}(\bar{S}.\bar{p})^2 + \bar{S}^2 = 1$$

- IN GENERAL

$$s^{\mu}_{(0)} = (0, \bar{\zeta}) \qquad \bar{\zeta} : \text{QUANTIZATION DIRECTION IN REST FRAME } (\bar{\zeta}^2 = 1)$$

$$s^{\mu} = \left(\frac{c}{E}\, \bar{S}.\bar{p}\, , \bar{S}\right) \qquad \text{e.g. } \bar{\zeta} = \bar{e}_z$$

WHERE
$$\bar{S} = \bar{\zeta} + \frac{(\bar{\zeta}.\bar{p})}{m_0(E + m_0 c^2)}\, \bar{p}$$

$$\bar{S}.\bar{p} = \bar{\zeta}.\bar{p} \left(\frac{E}{m_0 c^2}\right)$$

CHECK NORMALIZATION

$$-(S^0)^2 + \bar{S}^2 = -\frac{c^2}{E^2}(\bar{S}.\bar{p})^2 + \bar{S}^2$$

$$= -\frac{1}{m_0^2 c^2}(\bar{\zeta}.\bar{p})^2 + 1 + \frac{2(\bar{\zeta}.\bar{p})^2}{m_0(E + m_0 c^2)} + \frac{(\bar{\zeta}.\bar{p})^2 (E - m_0 c^2)}{m_0^2 c^2 \,(E + m_0 c^2)}$$

$$= 1 + \frac{(\bar{\zeta}.\bar{p})^2}{m_0^2 c^2 (E + m_0 c^2)}\left[-(E + m_0 c^2) + 2 m_0 c^2 + E - m_0 c^2\right]$$

$$\overset{!}{=} 1$$

$$\therefore \quad \boxed{s^{\mu} = \left(\frac{\bar{\zeta}.\bar{p}}{m_0 c}\, , \bar{\zeta} + \frac{(\bar{\zeta}.\bar{p})}{m_0(E + m_0 c^2)}\, \bar{p}\right)}$$

WITH $\bar{\zeta}$ REST FRAME QUANTIZATION AXIS

- EXAMPLE (LORENTZ BOOST ALONG Z-AXIS)

$$a^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & & \\ 0 & & 1 & \\ \sinh\omega & & & \cosh\omega \end{pmatrix}$$

$$p^{\mu} = a^{\mu}{}_{\nu}\, p^{\nu}_{(0)} \quad \text{WITH} \quad \beta = \frac{c\,|\bar{p}|}{E}, \quad \gamma = \frac{E}{m_0 c^2}$$

FOR $\bar{\zeta} = \bar{e}_z$

$$\begin{pmatrix} S^0 \\ S^1 \\ S^2 \\ S^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta\gamma \\ 0 \\ 0 \\ \gamma \end{pmatrix}$$

$$S^{\mu} = \left(\frac{p_z}{m_0 c}, \frac{E}{m_0 c^2}\, \bar{e}_z \right)$$

CONSISTENT WITH ABOVE FORMULA

FOR $\bar{\zeta} = \bar{e}_x$

$$S^{\mu} = \left(0, \bar{e}_x \right)$$

↳ SPINORS FOR PARTICLE WITH MOMENTUM ALONG Z-AXIS FROM REST FRAME SPINOR WITH QUANTIZATION DIRECTION $\bar{\zeta} = \bar{e}_z$

WHEN MOMENTUM DIRECTION & QUANTIZATION DIRECTION COINCIDE $\Rightarrow$ HELICITY λ

$$w(\bar{p}, \lambda \doteq \pm \tfrac{\hbar}{2}) = \underbrace{S(a)}_{\text{UNITARY MATRIX.}} w(\bar{0}, s_z^{(0)} = \pm \tfrac{\hbar}{2})$$

$$S(a) = \exp\left\{ + \frac{i}{2} \omega \, \sigma_{03} \right\}$$ FOR LORENTZ BOOST ALONG -Z DIRECTION

$$\sigma_{03} = - i \alpha_3$$

$$= \exp\left\{ \frac{\omega}{2} \alpha_3 \right\}$$

$$= \cosh\frac{\omega}{2} \, \mathbb{1}_{4\times4} + \sinh\frac{\omega}{2} \, \alpha_3$$

$$= \cosh\frac{\omega}{2} \begin{bmatrix} 1 & 0 & \tanh\frac{\omega}{2} & 0 \\ 0 & 1 & 0 & -\tanh\frac{\omega}{2} \\ \tanh\frac{\omega}{2} & 0 & 1 & 0 \\ 0 & -\tanh\frac{\omega}{2} & 0 & 1 \end{bmatrix}$$

FROM $\left. \begin{array}{l} \beta = \tanh\omega \\ \gamma = \cosh\omega \end{array} \right\}$

$$\tanh\frac{\omega}{2} = \frac{\tanh\omega}{1 + \sqrt{1 - \tanh^2\omega}}$$

$$= \frac{\beta}{1 + \sqrt{1 - \beta^2}} = \frac{c p_z}{E + m_0 c^2}$$

$$\cosh\frac{\omega}{2} = \frac{1}{\sqrt{1 - \tanh^2\frac{\omega}{2}}} = \sqrt{\frac{E + m_0 c^2}{2 m_0 c^2}}$$

$$S(a) = \sqrt{\frac{E + m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 & 0 & \frac{c\,p_z}{E+m_0c^2} & 0 \\ 0 & 1 & 0 & -\frac{c\,p_z}{E+m_0c^2} \\ \frac{c\,p_z}{E+m_0c^2} & 0 & 1 & 0 \\ 0 & -\frac{c\,p_z}{E+m_0c^2} & 0 & 1 \end{bmatrix}$$

$$= \sqrt{\frac{E + m_0c^2}{2m_0c^2}} \begin{bmatrix} \mathbb{1} & \frac{c\,\bar{\sigma}.\bar{p}}{E+m_0c^2} \\ \frac{c\,\bar{\sigma}.\bar{p}}{E+m_0c^2} & \mathbb{1} \end{bmatrix}$$

AS $\bar{p} = p_z\,\bar{e}_z$

e.g. $U(\bar{p}, \lambda = +\frac{\hbar}{2}) = \sqrt{\frac{E + m_0c^2}{2m_0c^2}}\, N \begin{bmatrix} \mathbb{1} & \frac{c\,\bar{\sigma}.\bar{p}}{E+m_0c^2} \\ \frac{c\,\bar{\sigma}.\bar{p}}{E+m_0c^2} & \mathbb{1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$= N \sqrt{\frac{E + m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 \\ 0 \\ \frac{c\,\bar{\sigma}.\bar{p}}{E+m_0c^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix}$$

$$= N \sqrt{\frac{E + m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 \\ 0 \\ \frac{c\,p_z}{E+m_0c^2} \\ 0 \end{bmatrix}$$

OK WITH RESULT (ON PAGE IV 20)

ANALOGOUSLY FOR OTHER HELICITY SPINORS.

- NORMALIZATION & COMPLETENESS OF DIRAC SPINORS

↳ DIRAC EQ.

$$\left(i\hbar\gamma^\mu \partial_\mu - m_0 c \right) \Psi(x,t) = 0$$

PARTICLE SOLUTION $\Psi(\bar{x},t) = e^{-\frac{i}{\hbar}p.x} \, U(\bar{p},\lambda)$

$$\left(\gamma^\mu p_\mu - m_0 c \right) U(\bar{p},\lambda) = 0$$

$$\underline{\underline{\left(\not{p} - m_0 c \right) U(\bar{p},\lambda) = 0}}$$

ANTI-PARTICLE SOLUTION $\Psi(\bar{x},t) = e^{+\frac{i}{\hbar}p.x} \, v(\bar{p},\lambda)$

$$\left(-\gamma^\mu p_\mu - m_0 c \right) v(\bar{p},\lambda) = 0$$

$$\underline{\underline{\left(\not{p} + m_0 c \right) v(\bar{p},\lambda) = 0}}$$

↳ NORMALIZATION OF HELICITY SPINORS

- $\bar{U}(\bar{p},\lambda)\, U(\bar{p},\lambda')$

$$= N^2 \left(\varphi_\lambda^+ \quad \varphi_\lambda^+ \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \right) \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi_{\lambda'} \\ \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \varphi_{\lambda'} \end{pmatrix}$$

$$= N^2 \left\{ \varphi_\lambda^+ \varphi_{\lambda'} - \varphi_\lambda^+ \frac{c^2 (\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})}{(E_{\bar{p}} + m_0 c^2)^2} \varphi_{\lambda'} \right\}$$

$$\bar{u}(\bar{p},\lambda)\, u(\bar{p},\lambda')$$

$$= N^2 \left\{ \varphi_\lambda^+ \varphi_{\lambda'} - \varphi_\lambda^+ \frac{c^2 \bar{p}^2}{(E_{\bar{p}} + m_0 c^2)^2} \varphi_{\lambda'} \right\}$$

$$c^2 \bar{p}^2 = E_{\bar{p}}^2 - m_0^2 c^4 = (E_{\bar{p}} - m_0 c^2)(E_{\bar{p}} + m_0 c^2)$$

$$= N^2 \left(\frac{2 m_0 c^2}{E_{\bar{p}} + m_0 c^2} \right) \underbrace{\varphi_\lambda^+ \varphi_{\lambda'}}_{\delta_{\lambda\lambda'}}$$

ORTHONORMALITY OF PAULI HELICITY SPINORS.

- $$u^+(\bar{p},\lambda)\, u(\bar{p},\lambda')$$

$$= N^2 \left(\frac{2 E_{\bar{p}}}{E_{\bar{p}} + m_0 c^2} \right) \delta_{\lambda\lambda'}$$

- DIFFERENT SPINOR NORMALIZATION CONVENTIONS

$$N = \sqrt{\frac{E_{\bar{p}} + m_0 c^2}{2 m_0 c^2}} \Rightarrow \begin{cases} \bar{u}(\bar{p},\lambda)\, u(\bar{p},\lambda') = \delta_{\lambda\lambda'} \\ u^+(\bar{p},\lambda)\, u(\bar{p},\lambda') = \dfrac{E_{\bar{p}}}{m_0 c^2} \delta_{\lambda\lambda'} \end{cases}$$

'NON-RELATIVISTIC' NORMALIZATION CHOICE

$$N = \sqrt{\frac{E_{\bar{p}} + m_0 c^2}{2 E_{\bar{p}}}} \Rightarrow \begin{cases} \bar{u}(\bar{p},\lambda)\, u(\bar{p},\lambda') = \dfrac{m_0 c^2}{E_{\bar{p}}} \delta_{\lambda\lambda'} \\ u^+(\bar{p},\lambda)\, u(\bar{p},\lambda') = \delta_{\lambda\lambda'} \end{cases}$$

COVARIANT NORMALIZATION CHOICE

$$N = \sqrt{E_{\bar{p}} + m_0 c^2} \Rightarrow \begin{cases} \bar{u}(\bar{p},\lambda)\, u(\bar{p},\lambda') = 2 m_0 c^2 \delta_{\lambda\lambda'} \\ u^+(\bar{p},\lambda)\, u(\bar{p},\lambda') = 2 E_{\bar{p}} \delta_{\lambda\lambda'} \end{cases}$$

- $\bar{v}(\bar{p},\lambda)\, v(\bar{p},\lambda')$

$$= N^2 \left\{ \varphi^+_{-\lambda} \frac{c^2(\bar{\sigma}.p)(\bar{\sigma}.\bar{p})}{(E_{\bar{p}}+m_0c^2)^2} \varphi_{-\lambda'} - \varphi^+_{-\lambda}\varphi_{-\lambda'} \right\}$$

$$= N^2\, \delta_{\lambda\lambda'} \left(\frac{-2m_0c^2}{E_{\bar{p}}+m_0c^2} \right)$$

- $v^+(\bar{p},\lambda)\, v(\bar{p},\lambda')$

$$= N^2\, \delta_{\lambda\lambda'} \left(\frac{2E_{\bar{p}}}{E_{\bar{p}}+m_0c^2} \right)$$

- DIFFERENT NORMALIZATION CONVENTIONS

$$N = \sqrt{\frac{E_{\bar{p}}+m_0c^2}{2m_0c^2}} \Rightarrow \begin{cases} \bar{v}(\bar{p},\lambda)\, v(\bar{p},\lambda') = -\delta_{\lambda\lambda'} \\ v^+(\bar{p},\lambda)\, v(\bar{p},\lambda') = \dfrac{E_{\bar{p}}}{m_0c^2}\,\delta_{\lambda\lambda'} \end{cases}$$

$$N = \sqrt{\frac{E_{\bar{p}}+m_0c^2}{2E_{\bar{p}}}} \Rightarrow \begin{cases} \bar{v}(\bar{p},\lambda)\, v(\bar{p},\lambda') = -\dfrac{m_0c^2}{E_{\bar{p}}}\,\delta_{\lambda\lambda'} \\ v^+(\bar{p},\lambda)\, v(\bar{p},\lambda') = \delta_{\lambda\lambda'} \end{cases}$$

$$N = \sqrt{E_{\bar{p}}+m_0c^2} \Rightarrow \begin{cases} \bar{v}(\bar{p},\lambda)\, v(\bar{p},\lambda') = -2m_0c^2\,\delta_{\lambda\lambda'} \\ v^+(\bar{p},\lambda)\, v(\bar{p},\lambda') = 2E_{\bar{p}}\,\delta_{\lambda\lambda'} \end{cases}$$

↳ COMPLETENESS

- $$\sum_{\lambda=\pm\frac{\hbar}{2}} U(\bar{p},\lambda)\,\bar{U}(\bar{p},\lambda) \qquad 4\times 4 \text{ MATRIX}$$

$$= N^2 \sum_\lambda \begin{pmatrix} \varphi_\lambda \\ \frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2}\varphi_\lambda \end{pmatrix} \begin{pmatrix} \varphi_\lambda^+ & -\varphi_\lambda^+ \frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2} \end{pmatrix}$$

$$= N^2 \left(\begin{array}{c|c} \sum_\lambda \varphi_\lambda\varphi_\lambda^+ & -\sum_\lambda \varphi_\lambda\varphi_\lambda^+ \frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2} \\ \hline \frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2}\sum_\lambda \varphi_\lambda\varphi_\lambda^+ & -\frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2}\sum_\lambda \varphi_\lambda\varphi_\lambda^+\frac{c\bar{\sigma}.\bar{p}}{E_p+m_0c^2} \end{array}\right)$$

$$\downarrow \quad \sum_{\lambda=\pm\frac{\hbar}{2}} \varphi_\lambda\varphi_\lambda^+ = \begin{pmatrix} \cos^2\theta/2 & e^{-i\phi}\sin\theta/2\cos\theta/2 \\ e^{i\phi}\sin\theta/2\cos\theta/2 & \sin^2\theta/2 \end{pmatrix}$$

$$+ \begin{pmatrix} \sin^2\theta/2 & -e^{-i\phi}\sin\theta/2\cos\theta/2 \\ e^{i\phi}\sin\theta/2\cos\theta/2 & \cos^2\theta/2 \end{pmatrix}$$

$$= \mathbb{1}_{2\times 2}.$$

$$= N^2 \begin{pmatrix} \mathbb{1} & -\frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2} \\ \frac{c\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m_0c^2} & -\frac{(E_{\bar{p}}-m_0c^2)}{(E_{\bar{p}}+m_0c^2)} \end{pmatrix}$$

$$= N^2 \frac{1}{E_{\bar{p}}+m_0c^2}\, c\,(\not{p} + m_0c)$$

- ANALOGOUSLY

$$\sum_{\lambda=\pm\frac{\hbar}{2}} v(\bar{p},\lambda)\,\bar{v}(\bar{p},\lambda)$$

$$= N^2 \begin{pmatrix} +\dfrac{E_{\bar{p}} - m_0c^2}{E_{\bar{p}} + m_0c^2} & \dfrac{-c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0c^2} \\ \dfrac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0c^2} & -\mathbb{1} \end{pmatrix} = +N^2 \frac{1}{E_{\bar{p}} + m_0c^2}\left(\not{p} - m_0c\right)c$$

- COMBINE BOTH

$$\sum_{\lambda=\pm\frac{\hbar}{2}} \left\{ u(\bar{p},\lambda)\,\bar{u}(\bar{p},\lambda) - v(\bar{p},\lambda)\,\bar{v}(\bar{p},\lambda) \right\}$$

$$= N^2 \left(\frac{2m_0c^2}{E_{\bar{p}} + m_0c^2} \right) \mathbb{1}_{4\times 4}$$

$$= 1 \qquad \text{FOR} \qquad N = \sqrt{\frac{E_{\bar{p}} + m_0c^2}{2m_0c^2}}$$

$$= \frac{m_0c^2}{E_{\bar{p}}} \qquad \text{FOR} \qquad N = \sqrt{\frac{E_{\bar{p}} + m_0c^2}{2E_{\bar{p}}}}$$

$$= 2m_0c^2 \qquad \text{FOR} \qquad N = \sqrt{E_{\bar{p}} + m_0c^2}$$

- ENERGY PROJECTION OPERATORS.

$$\hookrightarrow \boxed{\Lambda^{\pm}(\bar{p}) = \frac{\pm \not{p} + m_0 c}{2 m_0 c}} \qquad 4 \times 4 \quad \text{MATRIX}$$

BY USING $(\not{p} - m_0 c)\, u(\bar{p}, \lambda) = 0$

$(\not{p} + m_0 c)\, v(\bar{p}, \lambda) = 0$

$$\Lambda^{+}(\bar{p})\, u(\bar{p}, \lambda) = u(\bar{p}, \lambda)$$

$$\Lambda^{-}(\bar{p})\, u(\bar{p}, \lambda) = 0$$

$$\Lambda^{+}(\bar{p})\, v(\bar{p}, \lambda) = 0$$

$$\Lambda^{-}(\bar{p})\, v(\bar{p}, \lambda) = v(\bar{p}, \lambda)$$

$$\hookrightarrow \Lambda^{+}(\bar{p})\, \Lambda^{-}(\bar{p}) = \frac{1}{(2m_0 c)^2} \underbrace{(\not{p} + m_0 c)(-\not{p} + m_0 c)}_{-p^2 + m_0^2 c^2}$$

$$= 0$$

$$\hookrightarrow \left(\Lambda^{+}(\bar{p})\right)^2 = \frac{1}{(2m_0 c)^2} (\not{p} + m_0 c)(\not{p} + m_0 c)$$

$$= \frac{1}{(2m_0 c)^2} \left\{ \underbrace{p^2 + m_0^2 c^2}_{2 m_0^2 c^2} + 2 m_0 c \not{p} \right\}$$

$$= \frac{1}{2m_0 c} (\not{p} + m_0 c) = \Lambda^{+}(\bar{p})$$

↳ ANALOGOUSLY

$$\left(\Lambda^-(\bar{p})\right)^2 = \Lambda^-(\bar{p})$$

↳ $$\Lambda^+(\bar{p}) + \Lambda^-(\bar{p}) = \mathbb{1}_{4\times 4}$$

$\Lambda^{\pm}(p)$: PROJECTION OPERATORS PROJECTING OUT PARTICLE (+) OR ANTI-PARTICLE (−) SPINORS.

↳ FOR $N = \sqrt{E_{\bar{p}} + m_0 c^2}$ (COVARIANT NORMALIZATION)

$$\sum_{\lambda = \pm \frac{\hbar}{2}} u(\bar{p},\lambda)\,\bar{u}(\bar{p},\lambda) = (\not{p} + m_0 c)c = \Lambda^+(\bar{p})\, 2m_0 c^2$$

$$\sum_{\lambda = \pm \frac{\hbar}{2}} v(\bar{p},\lambda)\,\bar{v}(\bar{p},\lambda) = (\not{p} - m_0 c)c = -\Lambda^-(\bar{p})\, 2m_0 c^2$$

$$\Downarrow$$

$$\sum_{\lambda = \pm \frac{\hbar}{2}} \left\{ u(\bar{p},\lambda)\,\bar{u}(\bar{p},\lambda) - v(\bar{p},\lambda)\,\bar{v}(\bar{p},\lambda) \right\} = \left\{ \Lambda^+(\bar{p}) + \Lambda^-(\bar{p}) \right\} 2m_0 c^2$$

$$= 2m_0 c^2$$

(COMPLETENESS)

• HELICITY AND SPIN PROJECTION OPERATORS

↳ PARTICLE AT REST

SPIN QUANTIZATION AXIS $\bar{\xi}$ (e.g. $\bar{\xi} = \bar{e}_z$ Z-AXIS)

$$S^{(0)} \equiv (0, \bar{\xi})$$

$$\boxed{\Sigma(S^{(0)}) \equiv \frac{\mathbb{1} + \gamma_5 \not{S}^{(0)}}{2}}$$

$$= \frac{1}{2}\left(\mathbb{1} + \gamma_5 (-\bar{\gamma}.\bar{\xi})\right)$$

$$= \frac{1}{2}\left\{\begin{pmatrix} \mathbb{1}_{2\times2} & \\ & \mathbb{1}_{2\times2} \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\bar{\sigma}.\bar{\xi} \\ \bar{\sigma}.\bar{\xi} & 0 \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \frac{1}{2}(\mathbb{1} + \bar{\sigma}.\bar{\xi}) & 0 \\ 0 & \frac{1}{2}(\mathbb{1} - \bar{\sigma}.\bar{\xi}) \end{pmatrix}$$

$$= \frac{1}{2}\left\{\mathbb{1}_{4\times4} + (\bar{\Sigma}.\bar{\xi})\,\gamma_0\right\}$$

e.g. $\vec{\zeta} = \vec{e}_z$

$$\Sigma\left(S_z^{(0)}\right) = \frac{1}{2}\left(\mathbb{1}_{4\times 4} + \Sigma_3 \gamma_0\right)$$

$$\Sigma\left(S_z^{(0)}\right) U\left(\bar{0}, s_z^{(0)} = +\frac{\hbar}{2}\right) = \frac{1}{2}\left(\mathbb{1}_{4\times 4} + \gamma_0\right) N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = U\left(\bar{0}, s_z^{(0)} = +\frac{\hbar}{2}\right)$$

$$\Sigma\left(S_z^{(0)}\right) U\left(\bar{0}, s_z^{(0)} = -\frac{\hbar}{2}\right) = \frac{1}{2}\left(\mathbb{1}_{4\times 4} - \gamma_0\right) N \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0$$

$$\Sigma\left(\pm S_z^{(0)}\right) = \frac{1}{2}\left(\mathbb{1}_{4\times 4} \pm \Sigma_3 \gamma_0\right)$$

PROJECTS OUT SPINORS IN REST FRAME
WITH SPIN ALONG ± Z- AXIS.

NOTE • $\Sigma\left(S^{(0)}\right) + \Sigma\left(-S^{(0)}\right) = \mathbb{1}$

• $$\left(\Sigma\left(S^{(0)}\right)\right)^2 = \frac{1}{4}\left(\mathbb{1} + 2\gamma_5 \not{S}^{(0)} + \gamma_5 \not{S}^{(0)} \underbrace{\gamma_5 \not{S}^{(0)}}\right)$$

$$= \frac{1}{4}\left(\mathbb{1} + 2\gamma_5 \not{S}^{(0)} - \gamma_5 \underbrace{\left(\not{S}^{(0)}\right)^2}_{-1} \gamma_5\right)$$

$-\not{S}^{(0)} \gamma_5$

$$\left(\Sigma(S^{(0)})\right)^2 = \frac{1}{4}\left(\mathbb{1} + 2\gamma_5 \gamma^{(0)} + \underbrace{\gamma_5^2}_{\mathbb{1}}\right)$$

$$= \frac{1}{2}\left(\mathbb{1} + \gamma_5 \gamma^{(0)}\right)$$

$$= \Sigma(S^{(0)}) \qquad \text{(PROJECTION OPERATOR)}$$

↳ HELICITY PROJECTOR.

$$\Lambda_s = \frac{\bar{\Sigma}\cdot\bar{p}}{|\bar{p}|}$$

$$\boxed{\Sigma\left(p, \lambda = \pm\frac{\hbar}{2}\right) \equiv \frac{1}{2}\left(\mathbb{1} \pm \frac{\bar{\Sigma}\cdot\bar{p}}{|\bar{p}|}\right)}$$

$$\Sigma\left(p, +\frac{\hbar}{2}\right) + \Sigma\left(p, -\frac{\hbar}{2}\right) = \mathbb{1}$$

$$\left(\Sigma(p,\lambda)\right)^2 = \Sigma(p,\lambda)$$

$$\Sigma\left(p, \lambda = +\frac{\hbar}{2}\right) u\left(\bar{p}, \lambda = +\frac{\hbar}{2}\right) = 1$$

$$\Sigma\left(p, \lambda = -\frac{\hbar}{2}\right) u\left(\bar{p}, \lambda = -\frac{\hbar}{2}\right) = 1$$

$$\Sigma\left(p, \lambda = +\frac{\hbar}{2}\right) u\left(\bar{p}, \lambda = -\frac{\hbar}{2}\right) = 0$$

$$\Sigma\left(p, \lambda = -\frac{\hbar}{2}\right) u\left(\bar{p}, \lambda = +\frac{\hbar}{2}\right) = 0.$$

$$\Sigma\left(p, \lambda = +\frac{\hbar}{2}\right) v\left(\bar{p}, \lambda = +\frac{\hbar}{2}\right) = 0$$

$$\Sigma\left(p, \lambda = +\frac{\hbar}{2}\right) v\left(\bar{p}, \lambda = -\frac{\hbar}{2}\right) = v\left(\bar{p}, \lambda = -\frac{\hbar}{2}\right)$$

$$\Sigma\left(p, \lambda = -\frac{\hbar}{2}\right) v\left(\bar{p}, \lambda = -\frac{\hbar}{2}\right) = 0$$

$$\Sigma\left(p, \lambda = -\frac{\hbar}{2}\right) v\left(\bar{p}, \lambda = +\frac{\hbar}{2}\right) = v\left(\bar{p}, \lambda = +\frac{\hbar}{2}\right)$$

$\Sigma\left(p, \lambda = +\frac{\hbar}{2}\right)$ PROJECTS OUT POS. HELICITY PARTICLE
NEG. " ANTI-PARTICLE

$\Sigma\left(p, \lambda = -\frac{\hbar}{2}\right)$ VICE VERSA

↳ IMPORTANT SPECIAL CASE : MASSLESS PARTICLE ($m_0 = 0$)

$$E_{\bar{p}} = c|\bar{p}|$$

$$\not{p}\, u(\bar{p},\lambda) = 0$$

$$\gamma^0 u(\bar{p},\lambda) = \bar{\gamma}.\frac{\bar{p}}{|\bar{p}|}\, u(\bar{p},\lambda)$$

↓ MULTIPLY BY $\gamma_5\gamma^0$

$$\gamma_5\, u(\bar{p},\lambda) = \gamma_5\, \gamma^0 \bar{\gamma}.\frac{\bar{p}}{|\bar{p}|}\, u(\bar{p},\lambda)$$

$$= \frac{1}{|\bar{p}|}\begin{pmatrix} \bar{\sigma}.\bar{p} & 0 \\ 0 & \bar{\sigma}.\bar{p} \end{pmatrix} u(\bar{p},\lambda)$$

$$= \bar{\Sigma}.\bar{p}\;\; u(\bar{p},\lambda)$$

HELICITY PROJECTOR FOR MASSLESS PARTICLE

$$\boxed{\Sigma\left(\bar{p}, \lambda = \pm\frac{\hbar}{2}\right) = \frac{1}{2}\left(\mathbb{1} \pm \gamma_5\right) \quad \text{FOR } m_0 = 0}$$