IV

RELATIVISTIC FERMIONS

IV 1) DIRAC EQUATION

IV 2) LORENTZ COVARIANCE OF DIRAC EQUATION

IV 3) DIRAC PARTICLE IN CENTRAL POTENTIAL

IV 4) QUANTIZATION OF DIRAC FIELD

1) DIRAC EQUATION

- DERIVATION OF DIRAC EQ.

↳ SCHRÖDINGER EQ.

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

↳ DIRAC (1928) WAS LOOKING FOR A RELATIVISTIC COVARIANT VERSION OF SCHRÖDINGER EQ. <u>LINEAR</u> IN <u>SPACE-TIME</u> DERIVATIVES

$$\left[-i\hbar c\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]\Psi$$

$$= i\hbar \frac{\partial \Psi}{\partial t} = c\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + \hat{\beta} m_0 c^2$$

WITH $\hat{\vec{p}} = -i\hbar\vec{\nabla}$

$\hat{\alpha}_i$ CANNOT BE NUMBERS AS EQ. SHOULD BE INVARIANT UNDER SPATIAL ROTATIONS

↓

<u>TRY</u>: $\hat{\alpha}_i$ ARE $N \times N$ MATRICES, Ψ : $N \times 1$ COLUMN MATRIX

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

$$\underbrace{\left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau}}_{\hat{H}} \psi_\tau$$

$$= i\hbar \frac{\partial \psi_\sigma}{\partial t}$$

SUM OVER $\tau = 1 \dots N$ (SUMMATION CONVENTION)

NOTE : σ, τ ARE USUAL MATRIX INDICES !
WE DO NOT NEED TO MAKE DISTINCTION
BETWEEN COVARIANT / CONTRAVARIANT
AS IN CASE OF FOUR-VECTORS

↳ WE LIKE TO RECOVER RELATIVISTIC ENERGY-MOMENTUM RELATION FOR FREE PARTICLE

⇓

POSSIBLE IF EACH COMPONENT ψ_σ SATISFIES KLEIN-GORDON EQUATION

⇝ REQUIRE $\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2} \right) \psi_\sigma = 0$

PROOF

$$i\hbar \frac{\partial \psi_\sigma}{\partial t} = \left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau} \psi_\tau$$

$$\Downarrow \quad i\hbar \frac{\partial}{\partial t}$$

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau} i\hbar \frac{\partial \psi_\tau}{\partial t}$$

N × N MATRIX EQ.

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \left[-i\hbar c \sum_{i=1}^{3} \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0 c^2\right]$$

$$\cdot \left[-i\hbar c \sum_{j=1}^{3} \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0 c^2\right] \Psi$$

$$= -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \Psi}{\partial x^i \partial x^j}$$

$$- i\hbar m_0 c^3 \sum_{i=1}^{3} \left(\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i\right) \frac{\partial \Psi}{\partial x^i}$$

$$+ (m_0 c^2)^2 \hat{\beta}^2 \Psi$$

$\Downarrow$

WE WANT THIS TO BE IN FORM

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi + \frac{m_0^2 c^2}{\hbar^2} \Psi = 0$$

$\Downarrow$

REQUIRE:

$$\boxed{\begin{aligned} \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2 \delta_{ij} \mathbb{1}_{N \times N} \\ \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0 \\ \hat{\alpha}_i^2 = \hat{\beta}^2 &= \mathbb{1}_{N \times N} \end{aligned}}$$

↳ IN ADDITION : WE WANT $\hat{H}$ TO BE <u>HERMITIAN</u>

$$\Downarrow$$

$\hat{\alpha}_i, \hat{\beta}$ HAVE TO BE HERMITIAN

$$\boxed{\begin{aligned} \hat{\alpha}_i^+ &= \hat{\alpha}_i \\ \hat{\beta}^+ &= \hat{\beta} \end{aligned}}$$

(REAL EIGENVALUES)

- $\hat{\alpha}_i^2 = \hat{\beta}^2 = 1 \Rightarrow$ EIGENVALUES ARE ± 1

- $\hat{\alpha}_i = - \hat{\beta} \hat{\alpha}_i \hat{\beta}$

$$\Downarrow$$

$$\mathrm{Tr}\, \hat{\alpha}_i = - \mathrm{Tr} \left\{ \hat{\beta} \hat{\alpha}_i \hat{\beta} \right\} = - \mathrm{Tr} \left\{ \hat{\alpha}_i \right\}$$

$$\Downarrow$$

$$\left.\begin{aligned} &\mathrm{Tr}\, \hat{\alpha}_i = 0 \\ &\text{ANALOGOUSLY} \quad \mathrm{Tr}\, \hat{\beta} = 0 \end{aligned}\right\} \text{TRACELESS.}$$

- FOR $N = 2 \Rightarrow$ ONLY 3 ANTI-COMMUTING MATRICES POSSIBLE (PAULI - MATRICES)

- SINCE $\hat{\alpha}_i, \hat{\beta}$ HAVE EIGENVALUES ± 1 & ARE TRACELESS THEY MUST HAVE SAME NUMBER OF $+1$ AS -1 EIGENVALUES $\Rightarrow$ ONLY <u>EVEN</u> N ARE POSSIBLE

↳ SMALLEST POSSIBLE DIMENSION: $\underline{N = 4}$

$\hat{\alpha}_i, \hat{\beta}$ DIRAC MATRICES : (4 × 4 MATRICES)

⇓

DIRAC REPRESENTATION

$$\boxed{\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \qquad \hat{\beta}_i = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}}$$

WITH $\hat{\sigma}_i$ PAULI-MATRICES (2 × 2 MATRICES)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

NOTE $\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2\delta_{ij} \mathbb{1}_{2\times 2}$

⇝ $\hat{\alpha}_i, \hat{\beta}$ TRACELESS & HERMITIAN

⇝ CHECK OF ANTI-COMMUTATION RELATIONS ∇

$$\hat{\alpha}_i^2 = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{2\times 2} & \\ & \mathbb{1}_{2\times 2} \end{pmatrix} \overset{!}{=} \mathbb{1}_{4\times 4}$$

$$\hat{\beta}^2 \overset{!}{=} \mathbb{1}_{4\times 4}$$

$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} \overset{!}{=} 0$$

$$\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix}$$

$$\overset{!}{=} 2\delta_{ij} \cdot \mathbb{1}_{4\times 4}$$

⇝ WE CAN CHOOSE ANOTHER REPRESENTATION OF $\hat{\alpha}_i$, $\hat{\beta}$ BY <u>UNITARY</u> TF.

e.g. $\hat{\alpha}'_i = U \hat{\alpha}_i U^{-1}$

$$\hat{\beta}' = U \hat{\beta} U^{-1}$$

$$\begin{aligned}\hat{\alpha}'_i \hat{\alpha}'_j + \hat{\alpha}'_j \hat{\alpha}'_i &= U \hat{\alpha}_i U^{-1} U \hat{\alpha}_j U^{-1} \\ &\quad + U \hat{\alpha}_j U^{-1} U \hat{\alpha}_i U^{-1} \\ &= U \underbrace{\left(\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i \right)}_{2 \delta_{ij} \mathbb{1}_{4\times 4}} U^{-1} \\ &= 2 \delta_{ij} \underbrace{U U^{-1}}_{\mathbb{1}_{4\times 4}}\end{aligned}$$

⇓

PHYSICS DOES <u>NOT</u> DEPEND ON SPECIFIC REPRESENTATION OF DIRAC MATRICES $\hat{\alpha}_i$, $\hat{\beta}$

↳ ABOVE DIRAC REPRESENTATION IS OFTEN CONVENIENT AND USED IN MANY CASES

∴ DIRAC EQUATION (4×1 MATRIX EQ.)

$$\boxed{\left[- i\hbar c \, \hat{\vec{\alpha}} \cdot \vec{\nabla} + \hat{\beta} m_0 c^2 \right] \Psi = i\hbar \frac{\partial \Psi}{\partial t}}$$

$\hat{\vec{\alpha}} \equiv \left(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 \right)$ EACH $\hat{\alpha}_i$: 4×4 MATRIX

- CONTINUITY EQUATION

WE LIKE TO HAVE AN EQ. FOR 'PROBABILITY DENSITY'

$\Psi^+ \Psi$

⇝ DIRAC EQ.

$$-i\hbar c\, \hat{\alpha}_i \frac{\partial \Psi}{\partial x^i} + \hat{\beta}\, m_0 c^2 \Psi = i\hbar \frac{\partial \Psi}{\partial t} \qquad (*)$$

⇝ HERMITION CONJUGATE EQ.

$$+i\hbar c \frac{\partial \Psi^+}{\partial x^i}\, \hat{\alpha}_i + \Psi^+ \hat{\beta}\, m_0 c^2 = -i\hbar \frac{\partial \Psi^+}{\partial t} \qquad (**)$$

⇝ $\Psi^+ (*) - (**) \Psi$

$$i\hbar \left(\Psi^+ \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^+}{\partial t} \Psi \right)$$

$$= -i\hbar c \left(\Psi^+ \hat{\alpha}_i \frac{\partial \Psi}{\partial x^i} + \frac{\partial \Psi^+}{\partial x^i} \hat{\alpha}_i \Psi \right)$$

$$+ m_0 c^2 \cancel{\left(\Psi^+ \hat{\beta} \Psi - \Psi^+ \hat{\beta} \Psi \right)}$$

$$\Downarrow$$

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} . \bar{J} = 0 \qquad \text{CONTINUITY EQ.}$$

$\rho = \Psi^+ \Psi$	$\bar{J} = c\, \Psi^+ \hat{\alpha} \Psi$

ρ : PROBABILITY DENSITY
(POSITIVE DEFINITE)

$\bar{J}$: PROBABILITY CURRENT $\quad \bar{J} = (J^1, J^2, J^3)$

$$J^i \equiv c\,\Psi^+ \hat{\alpha}^i \Psi$$

- SOLUTIONS OF DIRAC EQ. FOR FREE PARTICLE

$\hookrightarrow \quad \Psi(\bar{x}, t) = \psi(\bar{x})\, e^{-\frac{i}{\hbar}Et}$

$$\left[-i\hbar c\ \hat{\bar{\alpha}}\cdot\bar{\nabla} + \hat{\beta}\, m_0 c^2\right]\psi(\bar{x}) = E\,\psi(\bar{x})$$

$\psi(\bar{x})$ is $\underbrace{4 \times 1 \quad \text{COLUMN MATRIX}}$
$\hookrightarrow$ SPINOR

$$\psi(\bar{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

WITH φ & χ
2×1 COLUMN MATRICES

$$\left[-i\hbar c \begin{pmatrix} 0 & \bar{\sigma}\cdot\bar{\nabla} \\ \bar{\sigma}\cdot\bar{\nabla} & 0 \end{pmatrix} + m_0 c^2 \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}\right]\begin{bmatrix} \varphi \\ \chi \end{bmatrix} = E \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

$$\begin{cases} -i\hbar c\ \bar{\sigma}\cdot\bar{\nabla}\,\chi + m_0 c^2\,\varphi = E\,\varphi \\ -i\hbar c\ \bar{\sigma}\cdot\bar{\nabla}\,\varphi - m_0 c^2\,\chi = E\,\chi \end{cases}$$

SOLUTIONS WITH GOOD MOMENTUM

$$\psi(\bar{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{+\frac{i}{\hbar}\bar{p}.\bar{x}}$$

WITH φ, χ INDEP. OF $\bar{x}$

$$\begin{cases} c\,\bar{\sigma}.\bar{p}\,\chi + m_o c^2 \varphi = E\varphi \\ c\,\bar{\sigma}.\bar{p}\,\varphi - m_o c^2 \chi = E\chi \end{cases}$$

2° EQ: $$\chi = \frac{1}{E + m_o c^2}\, c\,\bar{\sigma}.\bar{p}\,\varphi$$

$\downarrow$

IN 1° EQ. $$c\,\bar{\sigma}.\bar{p}\,\chi = (E - m_o c^2)\varphi$$

$$\Downarrow$$

$$\frac{c^2}{E + m_o c^2}(\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})\varphi = (E - m_o c^2)\varphi$$

$$\Downarrow$$

$$\frac{c^2}{E + m_o c^2}\bar{p}^2 = (E - m_o c^2)$$

$$c^2 \bar{p}^2 = E^2 - (m_o c^2)^2$$

$$\therefore \quad E^2 = c^2 p^2 + m_o^2 c^4$$

RELATIVISTIC ENERGY - MOMENTUM RELATION

'POSITIVE ENERGY' / PARTICLE SOLUTIONS

CAN BE WRITTEN IN FORM

$$\Psi(x,t) = e^{-\frac{i}{\hbar} p\cdot x}\, U_r(\bar{p})$$

$r = 1, 2$

U_r IS 4×1 SPINOR

WITH $p^\mu\left(\frac{E_{\bar p}}{c}, \bar p\right)$ $\quad x^\mu(ct, \bar x)$

$$E_{\bar p} = \sqrt{c^2 \bar p^2 + m_0^2 c^4}$$

$U_r(\bar p)$ IS SOLUTION OF

$$\left(c\, \bar\alpha\cdot\bar p + \beta m_0 c^2\right) U_r(\bar p) = E_{\bar p}\, U_r(\bar p)$$

WITH
$$U_r(\bar p) = \begin{pmatrix} \varphi_r \\ \dfrac{c\,\bar\sigma\cdot\bar p}{E_{\bar p} + m_0 c^2}\,\varphi_r \end{pmatrix} \qquad r = 1, 2$$

φ_r DENOTES 2-COMPONENT SPINOR (2×1 MATRIX)

2 INDEPENDENT SOLUTIONS ($r = 1, 2$)

- FOR PARTICLE AT <u>REST</u> ($\bar p = 0$)

$$\varphi_1 = N\begin{pmatrix}1\\0\end{pmatrix} \qquad \varphi_2 = N\begin{pmatrix}0\\1\end{pmatrix}$$

N: NORMALIZATION FACTOR

$$U_1(\bar 0) = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad U_2(\bar 0) = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

↳ 'NEGATIVE ENERGY' / ANTI-PARTICLE SOLUTIONS

CAN BE WRITTEN IN FORM

$$\underline{\underline{\Psi(\bar{x},t) = e^{+\frac{i}{\hbar}p.x}\, v_r(\bar{p})}}$$

$$r = 1, 2$$

v_r IS 4×1 SPINOR

WITH $p^\mu = \left(\frac{E_{\bar{p}}}{c}, \bar{p}\right)$ $\quad x^\mu (ct, \bar{x})$

$$E_{\bar{p}} = \sqrt{c^2\bar{p}^2 + m_0^2 c^4}$$

$v_r(\bar{p})$ IS SOLUTION OF

$$\left(- c\, \bar{\alpha}.\bar{p} + \beta m_0 c^2\right) v_r(\bar{p}) = - E_{\bar{p}}\, v_r(\bar{p})$$

DENOTE $\quad v_r(\bar{p}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$

$$\begin{cases} - c\, \bar{\sigma}.\bar{p}\, \chi + m_0 c^2 \varphi = - E_{\bar{p}}\, \varphi \\ - c\, \bar{\sigma}.\bar{p}\, \varphi - m_0 c^2 \chi = - E_{\bar{p}}\, \chi \end{cases}$$

FROM 1° EQ.

$$\varphi = \frac{1}{E_{\bar{p}} + m_0 c^2}\, c\, \bar{\sigma}.\bar{p}\, \chi$$

INSERT IN 2° EQ.

$$- \frac{c^2 \bar{p}^2}{E_{\bar{p}} + m_0 c^2}\, \chi = -\left(E_{\bar{p}} - m_0 c^2\right) \chi$$

↓

$$c^2 \bar{p}^2 = E_{\bar{p}}^2 - m_0^2 c^4$$

$$E_{\bar{p}}^2 = c^2 \bar{p}^2 + m_0^2 c^4$$

WE FIND AGAIN RELATIVISTIC ENERGY-MOMENTUM RELATION

$$v_r(\bar{p}) = \begin{pmatrix} \dfrac{c\,\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}} + m_0 c^2}\,\chi_r \\ \chi_r \end{pmatrix}$$

DESCRIBES ANTI-PARTICLE WITH MASS m_0,
MOMENTUM $\bar{p}$,
ENERGY $E_{\bar{p}}$

- FOR ANTI-PARTICLE AT REST ($\bar{p} = 0$)
WE MAKE OPPOSITE CHOICE (SEE LATER)

$$\chi_1 \equiv N \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi_2 \equiv N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_1(\bar{0}) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_2(\bar{0}) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- HELICITY

BOTH PARTICLE SOLUTION $u_r(\bar{p})$ AND
ANTI - PARTICLE SOLUTION $v_r(\bar{p})$
HAVE 2 - FOLD DEGENERACY $(r = 1, 2)$

↳ PARTICLE AT REST $(\bar{p} = 0)$

INTRODUCE OPERATOR $\bar{\Sigma} \equiv \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$

↑
4×4 MATRIX
WITH $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$
2 × 2 PAULI - MATR.

$\Sigma_3 \; u_1(\bar{0}) = u_1(\bar{0})$ "SPIN-UP" SOLUTION

$\Sigma_3 \; u_2(\bar{0}) = - u_2(\bar{0})$ "SPIN-DOWN" SOLUTION

SPIN - OPERATOR $\bar{S} \equiv \frac{\hbar}{2} \bar{\Sigma}$

$u_1(\bar{0})$ HAS SPIN $+\frac{\hbar}{2}$ ALONG Z - AXIS
$u_2(\bar{0})$ " " $-\frac{\hbar}{2}$ " " "

↳ ANALOGOUSLY FOR ANTI - PARTICLE AT REST

$\Sigma_3 \; v_1(\bar{0}) = - v_1(\bar{0})$

$\Sigma_3 \; v_2(\bar{0}) = + v_2(\bar{0})$

↳ FOR Z- COMPONENT OF $\bar{\Sigma}$ TO BE A GOOD QUANTUM NUMBER, IT HAS TO COMMUTE WITH $\hat{H}$

DIRAC EQ.

$$\left[-i\hbar c\,\bar{\alpha}\,\bar{\nabla} + \beta m_0 c^2\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

$$\downarrow \quad \hat{\bar{p}} \equiv -i\hbar\bar{\nabla}$$

$$\underbrace{\left[c\,\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2\right]}_{\hat{H}_D}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

DIRAC HAMILTONIAN

↳ $\left[\hat{H}_D, \Sigma_3\right]$ COMMUTATOR

AS BOTH β & Σ_3 ARE DIAGONAL

$$= \left[c\,\bar{\alpha}.\hat{\bar{p}}, \Sigma_3\right]$$

$$= c\left\{\begin{pmatrix} 0 & \bar{\sigma}.\bar{p} \\ \bar{\sigma}.\bar{p} & 0\end{pmatrix}\begin{pmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix} - \begin{pmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix}\begin{pmatrix} 0 & \bar{\sigma}.\bar{p} \\ \bar{\sigma}.\bar{p} & 0\end{pmatrix}\right\}$$

$$= c\begin{pmatrix} 0 & [\bar{\sigma}.\bar{p}, \sigma_3] \\ [\bar{\sigma}.\bar{p}, \sigma_3] & 0\end{pmatrix}$$

$$\downarrow \quad \text{USE} \quad [\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

$$= c\,2i\,\varepsilon_{i3k}\,p_i\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}$$

$$S_3 = \frac{\hbar}{2} \Sigma_3$$

$$[\hat{H}_D, \Sigma_3] = i\hbar c \begin{pmatrix} 0 & (\bar{\sigma} \times \bar{p})_3 \\ (\bar{\sigma} \times \bar{p})_3 & 0 \end{pmatrix}$$

$$= i\hbar c (\bar{\alpha} \times \bar{p})_3$$

$\neq 0$ FOR $\bar{p} \neq 0$ AND $\bar{p}$ NOT ALONG Z-AXIS

$\hat{H}_D$ AND Σ_3 DO NOT COMMUTE FOR $\bar{p} \neq 0$ AND $\bar{p}$ NOT ALONG Z-AXIS

<u>ONLY</u> FOR $\left.\begin{array}{l}\bar{p} = 0 \\ \text{OR } \bar{p} \text{ ALONG Z-AXIS}\end{array}\right\}$ SPIN PROJECTION ALONG Z-AXIS IS GOOD QUANTUM NUMBER !

↳ <u>HOW TO INTERPRET THE 2 SOLUTIONS FOR $\bar{p} \neq 0$</u> ?

CONSIDER HELICITY OPERATOR $\hat{\Lambda}_S \equiv \hat{S} \cdot \frac{\bar{p}}{|\bar{p}|}$

WITH $\hat{S} = \frac{\hbar}{2} \bar{\Sigma}$

$$\hat{\Lambda}_S = \frac{\hbar}{2} \frac{1}{|\bar{p}|} \begin{pmatrix} \bar{\sigma} \cdot \hat{p} & 0 \\ 0 & \bar{\sigma} \cdot \hat{p} \end{pmatrix}$$

COMMUTATOR WITH $\hat{H}_D$

$$\left[\hat{H}_D , \hat{\Lambda}_S \right]$$

$$= \left[c\,\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 , \frac{\hbar}{2} \bar{\Sigma}.\frac{\hat{\bar{p}}}{|\bar{p}|} \right]$$

$$= \frac{\hbar c}{2|\bar{p}|} \left[\bar{\alpha}.\hat{\bar{p}} , \bar{\Sigma}.\hat{\bar{p}} \right]$$

BECAUSE β AND $\bar{\Sigma}$ ARE BLOCK DIAGONAL MATRICES

$$= \frac{\hbar c}{2|\bar{p}|} \left\{ \begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma}.\hat{\bar{p}} & 0 \\ 0 & \bar{\sigma}.\hat{\bar{p}} \end{pmatrix} - \begin{pmatrix} \bar{\sigma}.\hat{\bar{p}} & 0 \\ 0 & \bar{\sigma}.\hat{\bar{p}} \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar c}{2|\bar{p}|} \left\{ \begin{pmatrix} 0 & (\bar{\sigma}.\hat{\bar{p}})^2 \\ (\bar{\sigma}.\hat{\bar{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\bar{\sigma}.\hat{\bar{p}})^2 \\ (\bar{\sigma}.\hat{\bar{p}})^2 & 0 \end{pmatrix} \right\}$$

$$= 0$$

$\therefore$ $\hat{H}_D$ AND $\hat{\Lambda}_S$ CAN BE DIAGONALIZED SIMULTANEOUSLY

$\downarrow$

SOLUTIONS OF DIRAC EQUATION ARE EIGENSTATES OF $\hat{\Lambda}_S$

$\hookrightarrow$ EIGENVALUE OF $\hat{\Lambda}_S$ (HELICITY OF PARTICLE) IS A GOOD QUANTUM NUMBER TO CHARACTERIZE SOLUTIONS OF DIRAC EQ.

HELICITY EIGENSTATE FOR PARTICLE SOLUTION

PARTICLE SOLUTION OF DIRAC EQ.

$$\Psi(\bar{x},t) = e^{-\frac{i}{\hbar}p.x} \; U_r(\bar{p})$$

WITH $$U_r(\bar{p}) = \begin{pmatrix} \varphi_r \\ \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2}\,\varphi_r \end{pmatrix} \qquad r = 1,2$$

$$\frac{\hat{\bar{S}}.\hat{\bar{p}}}{|\bar{p}|}\,\Psi(x,t)$$

$$= \frac{\hbar}{2}\frac{1}{|\bar{p}|}\,\bar{\Sigma}.(-i\bar{\nabla})\; e^{-\frac{i}{\hbar}p.x} \; U_r(\bar{p})$$

$$= \frac{\hbar}{2}\, e^{-\frac{i}{\hbar}p.x}\; \frac{\bar{\Sigma}.\bar{p}}{|\bar{p}|}\; U_r(\bar{p})$$

NOTE: $\bar{p}$ (WITHOUT HAT ^ IS EIGENVALUE OF OPERATOR $\hat{\bar{p}}$)

$$\bar{\Sigma}.\frac{\bar{p}}{|\bar{p}|}\,U_r(\bar{p}) = \frac{1}{|\bar{p}|}\begin{pmatrix} \bar{\sigma}.\bar{p} & \\ & \bar{\sigma}.\bar{p} \end{pmatrix}\begin{pmatrix} \varphi_r \\ \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2}\,\varphi_r \end{pmatrix}$$

$$= \frac{1}{|\bar{p}|}\begin{pmatrix} \bar{\sigma}.\bar{p}\;\varphi_r \\ c\,\frac{(\bar{\sigma}.\bar{p})^2}{E_p + m_0 c^2}\,\varphi_r \end{pmatrix}$$

$\downarrow$

CHOOSE φ_r SUCH THAT

$$\frac{\bar{\sigma}\cdot\bar{p}}{|\bar{p}|}\,\varphi_r = \pm\,\varphi_r \qquad \begin{matrix} + \text{ FOR } r=1 \\ - \text{ FOR } r=2 \end{matrix}$$

$$\Downarrow$$

$$\frac{\bar{\Sigma}\cdot\bar{p}}{|\bar{p}|}\,U_r(\bar{p}) = \pm\begin{pmatrix} \varphi_r \\ \frac{c\,\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m_0c^2}\,\varphi_r \end{pmatrix} = \pm\,U_r(\bar{p})$$

$$\Downarrow$$

$$\frac{\bar{S}\cdot\hat{\bar{p}}}{|\bar{p}|}\,\Psi(\bar{x},t) = \left(\pm\frac{\hbar}{2}\right)\Psi(\bar{x},t)$$

$\Psi(\bar{x},t)$ IS EIGENSTATE OF $\hat{\Lambda}_s$
WITH EIGENVALUE $\pm\frac{\hbar}{2}$

WE DENOTE EIGENVALUE AS $\lambda = \pm\frac{\hbar}{2}$
SPIN PROJECTION ALONG DIRECTION OF MOMENTUM

$$\frac{\bar{S}\cdot\hat{\bar{p}}}{|\bar{p}|}\,\Psi(\bar{x},t) = \lambda\,\Psi(\bar{x},t)$$

↳ FOR ARBITRARY DIRECTION

$$\frac{\bar{p}}{|\bar{p}|} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

↳ POSITIVE HELICITY STATE $\left(\lambda = +\frac{\hbar}{2}\right)$

CORRESPONDS WITH $r = 1$

$$\bar{\sigma}\cdot\frac{\bar{p}}{|\bar{p}|}\,\varphi_{\lambda=+\frac{\hbar}{2}} = +\,\varphi_{\lambda=+\frac{\hbar}{2}}$$

↳ DENOTE STATE BY ITS EIGENVALUE $\left(\text{OF } \frac{\hbar}{2}\,\bar{\sigma}\cdot\frac{\bar{p}}{|\bar{p}|}\right)$

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\varphi_{\lambda=+\frac{\hbar}{2}}}$$

$$a\cos\theta + b\sin\theta\, e^{-i\phi} = a$$

$$b\sin\theta\, e^{-i\phi} = a\, 2\sin^2\theta/2$$

$$b\cos\theta/2\, e^{-i\phi} = a\sin\theta/2$$

$$\varphi_{\lambda=+\frac{\hbar}{2}} = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

CORRESPONDS TO NORMALIZATION CHOICE

$$\varphi^{+}_{\lambda=+\frac{\hbar}{2}}\,\varphi_{\lambda=+\frac{\hbar}{2}} = 1.$$

↳ NEGATIVE HELICITY STATE $(\lambda = -\frac{\hbar}{2})$

CORRESPONDS WITH $r = 2$

$$\frac{\bar{\sigma}.\bar{p}}{|\bar{p}|}\,\varphi_{\lambda=-\frac{\hbar}{2}} = -\,\varphi_{\lambda=-\frac{\hbar}{2}}$$

ANALOGOUSLY

$$\varphi_{\lambda=-\frac{\hbar}{2}} = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$\varphi_\lambda^\dagger\,\varphi_{\lambda'} = \delta_{\lambda\lambda'}$$: ORTHONORMALITY CONDITION

↳ DIRAC PARTICLE SPINOR SOLUTIONS CAN BE DENOTED AS

$U(\bar{p},\lambda)$ WITH $\lambda = +\frac{\hbar}{2} \longleftrightarrow r = 1$

$\lambda = -\frac{\hbar}{2} \longleftrightarrow r = 2$

$$U(\bar{p},\lambda) = N \begin{pmatrix} \varphi_\lambda \\ \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2}\,\varphi_\lambda \end{pmatrix}, \text{ WITH } \lambda = \pm\frac{\hbar}{2}$$

$$\frac{\hat{\bar{S}}.\bar{p}}{|\bar{p}|}\,U(\bar{p},\lambda) = \lambda\,U(\bar{p},\lambda)$$ HELICITY EIGENSTATES

N IS NORMALIZATION FACTOR (SEE FURTHER)

↳ HELICITY EIGENSTATE FOR ANTI-PARTICLE SOLUTION

ANTI-PARTICLE SOLUTION OF DIRAC EQ.

$$\Psi(\bar{x},t) = e^{+\frac{i}{\hbar} p.x} \, v_r(\bar{p})$$

WITH $v_r(\bar{p}) = \begin{pmatrix} \frac{c\,\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \chi_r \\ \chi_r \end{pmatrix}$ FOR $r = 1, 2$

$$\hookrightarrow \quad \hat{S}.\frac{\hat{p}}{|\bar{p}|} \Psi(\bar{x},t) = -\frac{\hbar}{2} e^{+\frac{i}{\hbar} p.x} \frac{\bar{\Sigma}.\bar{p}}{|\bar{p}|} v_r(\bar{p})$$

CHOOSE

$$\hat{S}.\frac{\bar{p}}{|\bar{p}|} v(\bar{p},\lambda) = -\lambda \, v(\bar{p},\lambda)$$

WITH $\lambda = +\frac{\hbar}{2}$, $r = 1$

$\lambda = -\frac{\hbar}{2}$, $r = 2$

NOTE: FOR ANTI-PARTICLE SPINOR

OPPOSITE SIGN FOR v !

↓

$$\hat{S}.\frac{\hat{p}}{|\bar{p}|} \Psi(\bar{x},t) = \lambda \, \Psi(\bar{x},t)$$

$$\bar{\Sigma} \cdot \frac{\bar{p}}{|\bar{p}|} \, v(\bar{p}, \lambda = \pm \tfrac{\hbar}{2}) = \mp \; v(\bar{p}, \lambda = \pm \tfrac{\hbar}{2})$$

$$\frac{1}{|\bar{p}|} \begin{pmatrix} \bar{\sigma}.\bar{p} & \\ & \bar{\sigma}.\bar{p} \end{pmatrix} \begin{pmatrix} c \dfrac{\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \chi_r \\ \chi_r \end{pmatrix} = \mp \begin{pmatrix} c \dfrac{\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \chi_r \\ \chi_r \end{pmatrix}$$

SATISFIED FOR

$$\bar{\sigma} \frac{\bar{p}}{|\bar{p}|} \chi_1 = - \chi_1$$

$$\bar{\sigma} . \frac{\bar{p}}{|\bar{p}|} \chi_2 = + \chi_2$$

$$\chi_1 = \varphi_{\lambda = -\frac{\hbar}{2}} = \begin{pmatrix} - e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$$\chi_2 = \varphi_{\lambda = +\frac{\hbar}{2}} = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix}$$

$\therefore$ DIRAC ANTI-PARTICLE SPINORS CAN BE DENOTED AS

$$v(\bar{p}, \lambda) = N \begin{pmatrix} \dfrac{c \, \bar{\sigma}.\bar{p}}{E_{\bar{p}} + m_0 c^2} \varphi_{-\lambda} \\ \varphi_{-\lambda} \end{pmatrix}$$

$$\hat{\bar{S}} . \frac{\bar{p}}{|\bar{p}|} \, v(\bar{p}, \lambda) = - \lambda \; v(\bar{p}, \lambda)$$

NON - RELATIVISTIC CORRESPONDENCE OF DIRAC EQ.

⤳ CONSIDER NON-REL. PARTICLE ($v/c \ll 1$) WITH CHARGE e MOVING IN APPLIED ELECTROMAGNETIC FIELD $A^{\mu}(\Phi, \bar{A})$

⇓

DESCRIBED BY 'MINIMAL SUBSTITUTION' (SEE ABOVE)

$$\hat{\bar{p}} \to \hat{\bar{p}} - \frac{e}{c}\bar{A}$$

& ELECTROSTATIC POTENTIAL $e\underline{\Phi}$

$$\underbrace{\left[c\,\hat{\bar{\alpha}}\cdot\left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right) + \hat{\beta}\, m_0 c^2 + e\underline{\Phi}\right]}_{\hat{H}}\underline{\Psi} = i\hbar\frac{\partial \underline{\Psi}}{\partial t}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}} + \beta m_0 c^2 \qquad \text{FREE DIRAC HAM.}$$

$$\hat{H}_1 = -e\,\hat{\bar{\alpha}}\cdot\bar{A} + e\underline{\Phi}$$

⤳ NON - REL. REDUCTION OF DIRAC EQ.

$$\underline{\Psi}(\bar{x},t) = e^{-\frac{i}{\hbar} m_0 c^2 t}\begin{bmatrix}\varphi \\ \chi\end{bmatrix}$$

⇑

* FOR POSITIVE ENERGY SOLUTIONS:
* KINETIC ENERGY APPROXIMATED RELATIVE TO REST MASS

$$i\hbar \frac{\partial \Psi}{\partial t} = (m_0 c^2)\, \Psi + e^{-\frac{i}{\hbar} m_0 c^2 t} \left(i\hbar \frac{\partial}{\partial t}\right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\Downarrow$$

$$\left(c\, \hat{\bar{\alpha}} \cdot \hat{\bar{\pi}} + \beta m_0 c^2 + e\,\Phi \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m_0 c^2 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

WITH $\hat{\bar{\pi}} \equiv \hat{\bar{p}} - \frac{e}{c} \bar{A}$

$$\Downarrow$$

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c \begin{pmatrix} \bar{\sigma} \cdot \hat{\bar{\pi}}\, \chi \\ \bar{\sigma} \cdot \hat{\bar{\pi}}\, \varphi \end{pmatrix} - 2 m_0 c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix} + e\,\Phi \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\begin{cases} i\hbar \dfrac{\partial \varphi}{\partial t} = c\, \bar{\sigma} \cdot \hat{\bar{\pi}}\, \chi + e\,\Phi\, \varphi \\ i\hbar \dfrac{\partial \chi}{\partial t} = c\, \bar{\sigma} \cdot \hat{\bar{\pi}}\, \varphi - 2 m_0 c^2 \chi + e\,\Phi\, \chi \end{cases}$$

$\rightsquigarrow$ 2° EQ. : NEGLECT KINETIC ENERGY $i\hbar \frac{\partial \chi}{\partial t}$ CONTRIBUTION

& POTENTIAL ENERGY $e\,\Phi\,\chi$ CONTRIBUTION

RELATIVE TO REST MASS TERM $-2 m_0 c^2 \chi$

$$\rightsquigarrow \quad \chi \simeq \frac{\bar{\sigma} \cdot \hat{\bar{\pi}}}{2 m_0 c}\, \varphi$$

⇝ 1° EQ. :

$$i\hbar \frac{\partial \varphi}{\partial t} \simeq c\left(\bar{\sigma}\cdot\hat{\bar{\Pi}}\right)\frac{\bar{\sigma}\cdot\hat{\bar{\Pi}}}{2m_0 c}\varphi + e\Phi\varphi$$

USE $\left(\bar{\sigma}\cdot\hat{\bar{\Pi}}\right)\left(\bar{\sigma}\cdot\hat{\bar{\Pi}}\right) = \hat{\bar{\Pi}}\cdot\hat{\bar{\Pi}} + i\,\bar{\sigma}\cdot\left(\hat{\bar{\Pi}}\times\hat{\bar{\Pi}}\right)$

$$\hat{\bar{\Pi}}\cdot\hat{\bar{\Pi}} = \left(\hat{\bar{P}} - \frac{e}{c}\bar{A}\right)\cdot\left(\hat{\bar{P}} - \frac{e}{c}\bar{A}\right)$$

$$= \hat{\bar{P}}^2 - \frac{e}{c}\left(\hat{\bar{P}}\cdot\bar{A} + \bar{A}\cdot\hat{\bar{P}}\right) + \frac{e^2}{c^2}\bar{A}^2$$

$$\hat{\bar{\Pi}}\times\hat{\bar{\Pi}} = \left(\hat{\bar{P}} - \frac{e}{c}\bar{A}\right)\times\left(\hat{\bar{P}} - \frac{e}{c}\bar{A}\right)$$

$$= -\frac{e}{c}\left(\hat{\bar{P}}\times\bar{A} + \bar{A}\times\hat{\bar{P}}\right)$$

$$= i\hbar\frac{e}{c}\left(\bar{\nabla}\times\bar{A}\right)$$

$$= i\frac{e\hbar}{c}\bar{B}$$

⇓

$$\boxed{i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{\left(\hat{\bar{P}} - \frac{e}{c}\bar{A}\right)^2}{2m_0} - \frac{e\hbar}{2m_0 c}\bar{\sigma}\cdot\bar{B} + e\Phi\right]\varphi}$$

PAULI EQUATION

↳ DESCRIBES NON-REL. PARTICLE OF SPIN 1/2
MOVING IN AN ELECTROMAGNETIC FIELD

2° TERM : $-\bar{\mu}\cdot\bar{B}$

WITH $\bar{\mu}$: MAGNETIC MOMENT $\bar{\mu} = \frac{e}{2m_0 c}g\bar{S}$

SPIN $\bar{S} = \frac{\hbar}{2}\bar{\sigma}$, $g = 2$: GYROMAGNETIC RATIO