I.

NON-RELATIVISTIC QUANTUM THEORY OF MANY-PARTICLE SYSTEMS

I 1) BOSONS & FERMIONS

I 2) FREE ELECTRON GAS

I 3) SECOND QUANTIZATION

↳ BOSONS

↳ FERMIONS

↳ HOMOGENEOUS ELECTRON GAS (JELLIUM MODEL)

1) BOSONS & FERMIONS

↳ 2 PARTICLE STATE : ↗ ONE PARTICLE IN STATE a : ψ_a
↘ SECOND PARTICLE IN STATE b : ψ_b

* CLASSICAL PHYSICS : PARTICLES CAN BE DISTINGUISHED

$$\psi(\bar{r}_1, \bar{r}_2) = \psi_a(\bar{r}_1)\, \psi_b(\bar{r}_2)$$

* IN QUANTUM PHYSICS : PARTICLES ARE INDISTINGUISHABLE

2 WAYS TO TAKE CARE OF THIS

$$\psi_{\pm}(\bar{r}_1, \bar{r}_2) = \underset{\uparrow \text{ NORMALIZATION}}{A} \left[\psi_a(\bar{r}_1)\, \psi_b(\bar{r}_2) \pm \psi_b(\bar{r}_1)\, \psi_a(\bar{r}_2) \right]$$

$+$: WAVE FUNCTION IS SYMMETRIC w.r.t. INTERCHANGE $1 \leftrightarrow 2$: <u>BOSONS</u>

$-$: WAVE FUNCTION IS ANTI-SYMMETRIC w.r.t INTERCHANGE $1 \leftrightarrow 2$: <u>FERMIONS</u>

↳ 'DEEP CONNECTION' (RELATIVISTIC QUANTUM FIELD THEORY) BETWEEN SYMMETRY OF WF & SPIN

SPIN - STATISTICS THEOREM
BOSONS (SYMM. UNDER $1 \leftrightarrow 2$) HAVE INTEGER SPIN
FERMIONS (ANTISYMM UNDER $1 \leftrightarrow 2$) HAVE HALF INTEGER SPIN

↳ PAULI EXCLUSION PRINCIPLE

2 FERMIONS CANNOT OCCUPY SAME STATE $\Longrightarrow$!

IF $\psi_b = \psi_a$

$$\psi_-(\bar{r}_1, \bar{r}_2) = A \cdot \left[\psi_a(\bar{r}_1)\psi_a(\bar{r}_2) - \psi_a(\bar{r}_1)\psi_a(\bar{r}_2)\right]$$

$$= 0$$

↳ EXCHANGE OPERATOR P

$$\underline{P}\,\psi(\bar{r}_1, \bar{r}_2) = \psi(\bar{r}_2, \bar{r}_1)$$

$$\underline{P}^2 = 1$$

FOR 2 IDENTICAL PARTICLES $\left(m_1 = m_2,\ V(\bar{r}_1, \bar{r}_2) = V(\bar{r}_2, \bar{r}_1)\right)$

$$\Downarrow$$

$$[\underline{P}, H] = 0$$

$\therefore$ EIGENSTATES OF H ARE EITHER SYMMETRIC OR ANTISYMM w.r.t $1 \leftrightarrow 2$

$$\boxed{\psi_\pm(\bar{r}_1, \bar{r}_2) = \pm\, \psi_\pm(\bar{r}_2; \bar{r}_1)}$$

$+ \rightarrow$ BOSONS

$- \rightarrow$ FERMIONS

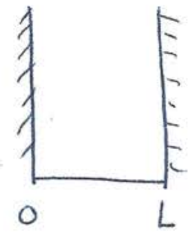
• EXCHANGE FORCES

2 PARTICLES IN 1 DIM $\quad \psi_a(x)$ STATE a

$\psi_b(x)$ STATE b

$a \neq b$

e.g.



↳ DISTINGUISHABLE: $\psi(x_1, x_2) = \psi_a(x_1)\,\psi_b(x_2)$

↳ BOSONS: $\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right]$

↳ FERMIONS: $\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)\right]$

WHAT IS $\langle\ \rangle$ OF $(x_1 - x_2)^2$ DISTANCE^2 BETWEEN 1&2

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

↳ DISTINGUISHABLE

$$\langle x_1^2 \rangle = \int dx_1\, x_1^2 |\psi_a(x_1)|^2 \underbrace{\int dx_2\, |\psi_b(x_2)|^2}_{1}$$

$$= \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \langle x \rangle_a \langle x \rangle_b$$

↳ IDENTICAL PARTICLES

$$\langle x_1^2 \rangle = \frac{1}{2} \Big\{ \int dx_1 \; x_1^2 \; |\psi_a(x_1)|^2 \overbrace{\int dx_2 \; |\psi_b(x_2)|^2}^{1}$$

$$+ \int dx_1 \; x_1^2 \; |\psi_b(x_1)|^2 \int dx_2 \; |\psi_a(x_2)|^2$$

$$\pm \int dx_1 \; x_1^2 \; \psi_a^*(x_1)\, \psi_b(x_1) \int dx_2 \; \psi_b^*(x_2)\, \psi_a(x_2)$$

$$\pm \int dx_1 \; x_1^2 \; \psi_b^*(x_1)\, \psi_a(x_1) \underbrace{\int dx_2 \; \psi_a^*(x_2)\, \psi_b(x_2)}_{0}$$

$$= \frac{1}{2} \Big\{ \langle x^2 \rangle_a + \langle x^2 \rangle_b + 0 + 0 \Big\}$$

$$\langle x_2^2 \rangle = \langle x_1^2 \rangle \qquad \text{IDENTICAL !}$$

$$\langle x_1 x_2 \rangle = \frac{1}{2} \Big\{ 2 \langle x \rangle_a \langle x \rangle_b$$

$$\pm \int dx_1 \; x_1 \; \psi_a^*(x_1)\, \psi_b(x_1) \int dx_2 \; x_2 \; \psi_b^*(x_2)\, \psi_a(x_2)$$

$$\pm \underbrace{\int dx_1 \; x_1 \; \psi_b^*(x_1)\, \psi_a(x_1)}_{\langle x \rangle_{ba}} \underbrace{\int dx_2 \; x_2 \; \psi_a^*(x_2)\, \psi_b(x_2)}_{\langle x \rangle_{ab}}$$

$$\langle x_1 x_2 \rangle = \langle x \rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2$$

∴ DISTINGUISHABLE PARTICLES

$$\langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b$$

∴ IDENTICLE PARTICLES.

$$\langle (x_1 - x_2)^2 \rangle_\pm = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b \mp 2 \left| \langle x \rangle_{ab} \right|^2$$

$$\boxed{\langle (x_1 - x_2)^2 \rangle_\pm = \langle (x_1 - x_2)^2 \rangle_d \mp 2 \left| \langle x \rangle_{ab} \right|^2}$$

INDUCES A CORRELATION BETWEEN PARTICLES. (EXCHANGE 'FORCE')

→ 'ATTRACTION' FOR BOSONS

PULLS THEM CLOSER TOGETHER

$\langle (x_1 - x_2)^2 \rangle$ DECREASES

→ 'REPULSIVE' FOR FERMIONS

PUSHES THEM FURTHER APART

$\langle (x_1 - x_2)^2 \rangle$ INCREASES

TOTAL WF INCLUDES SPIN ! → COVALENT BOND

2) FREE ELECTRON GAS (SOMMERFELD)

↳ FREE PARTICLES (e^-) IN A BOX

$$V(x,y,z) = \begin{cases} 0 & , \; 0 < x < \ell_x \;\&\; 0 < y < \ell_y \;\&\; 0 < z < \ell_z \\ \infty & \text{OTHERWISE} \end{cases}$$

(3D ANALOGON OF INFINITE SQUARE WELL POT.)

↳ $-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi$ WITHIN BOX

$$\psi(x,y,z) = X(x)\, Y(y)\, Z(z)$$

$$\begin{cases} -\frac{\hbar^2}{2m} \nabla^2 X = E_x X \\ -\frac{\hbar^2}{2m} \nabla^2 Y = E_y Y \\ -\frac{\hbar^2}{2m} \nabla^2 Z = E_z Z \end{cases} \qquad E = E_x + E_y + E_z$$

$$k_x \equiv \frac{\sqrt{2m E_x}}{\hbar} \;, \quad k_y \equiv \frac{\sqrt{2m E_y}}{\hbar} \;, \quad k_z \equiv \frac{\sqrt{2m E_z}}{\hbar}$$

SOLUTIONS

$$\begin{cases} X(x) = A_x \sin(k_x x) + B_x \cos(k_x x) \\ Y(y) = A_y \sin(k_y y) + B_y \cos(k_y y) \\ Z(z) = A_z \sin(k_z z) + B_z \cos(k_z z) \end{cases}$$

↳ BOUNDARY CONDITION

$\rightsquigarrow \quad X(0) = Y(0) = Z(0) = 0$

$\Rightarrow \quad B_x = B_y = B_z = 0$

$\rightsquigarrow \quad X(\ell_x) = Y(\ell_y) = Z(\ell_z) = 0$

$$\Rightarrow \quad k_x = \frac{n_x \pi}{\ell_x}, \quad k_y = \frac{n_y \pi}{\ell_y}, \quad k_z = \frac{n_z \pi}{\ell_z}$$

$$n_x, n_y, n_z = 1, 2, 3, \ldots$$

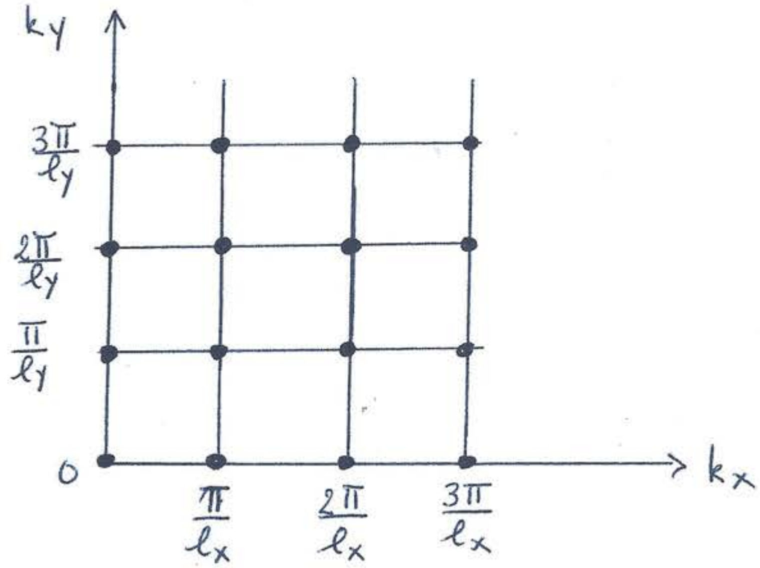
$$\therefore \quad \psi_{n_x n_y n_z} = C \sin\left(\frac{n_x \pi}{\ell_x} x\right) \sin\left(\frac{n_y \pi}{\ell_y} y\right) \sin\left(\frac{n_z \pi}{\ell_z} \pi\right)$$

NORMALIZATION $\quad C = \sqrt{\dfrac{8}{\ell_x \ell_y \ell_z}}$

$$E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_x^2}{\ell_x^2} + \frac{n_y^2}{\ell_y^2} + \frac{n_z^2}{\ell_z^2} \right) = \frac{\hbar^2 \bar{k}^2}{2m}$$

WAVEVECTOR $\quad \bar{k} = (k_x, k_y, k_z)$

RECIPROCAL SPACE (IN 3D)



EACH POINT IN GRID CORRESPONDS TO A STATIONARY STATE

↓

OCCUPIES A VOLUME (CELL) IN RECIPROCAL SPACE — "k - SPACE"

$$\frac{\pi^3}{\ell_x \ell_y \ell_z} = \frac{\pi^3}{V}$$ INVERSELY PROPORTIONA[L] TO PHYSICAL VOLUME

- N ATOMS
  EACH ATOM: q FREE e^- $(q = 1, 2)$
  $\therefore$ TOTAL Nq FREE e^-

Nq FREE e^- IN BOX

WILL FILL LEVELS ACCORDING TO PAULI PRINCIPLE

2 (BOTH SPIN ORIENTATIONS) CAN OCCUPY A STATIONARY STATE

$\Downarrow$

e^- OCCUPY A FINITE VOLUME IN k-SPACE

ONE OCTANT OF SPHERE OF RADIUS k_F $\Rightarrow$ FERMI SURFACE

$$\frac{1}{8}\,\frac{4}{3}\pi k_F^3 = \underbrace{\frac{Nq}{2}}_{\substack{\downarrow \\ \#\ e^-\ \text{PAIRS OF}\ \uparrow\downarrow \\ \text{THAT OCCUPY} \\ \text{STATIONARY STATES}}} \cdot \left(\frac{\pi^3}{V}\right)$$

$\hookrightarrow$ VOLUME IN k-SPACE OCCUPIED BY 1 STATIONARY STATE

$\Downarrow$

$$\boxed{k_F = (3\rho\pi^2)^{1/3}}$$

$\rightarrow$ DETERMINES FERMI SURFACE

$$\rho = \frac{Nq}{V}$$

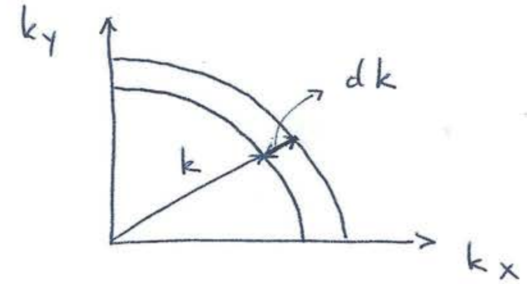
FREE e^- DENSITY (# FREE e^- / UNIT VOLUME)

FERMI ENERGY : ENERGY OF MOST ENERGETIC e^-

$$\boxed{E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}}$$

e.g. Cu : $E_F \simeq 7.\,\text{eV}$

↳ TOTAL ENERGY OF e^- GAS



⇝ # e^- IN SHELL k, $k+dk$

SPIN → $$\frac{2\left[\frac{1}{8}\left(4\pi k^2\right) dk\right]}{\left[\frac{\pi^3}{V}\right]} = \frac{V}{\pi^2} k^2 dk$$

↳ VOLUME IN k-SPACE OF 1 STATE

⇝ ENERGY OF SHELL k, $k+dk$

$$dE = \frac{\hbar^2 k^2}{2m} \frac{V}{\pi^2} k^2 dk$$

⇝ TOTAL ENERGY

$$E_{tot} = \frac{\hbar^2 V}{2m\pi^2} \int_0^{k_F} dk \; k^4$$

$$\boxed{E_{tot} = \frac{\hbar^2 V}{10\pi^2 m} k_F^5} = \frac{\hbar^2 \left(3\pi^2 N q\right)^{5/3}}{10\,\pi^2 m} V^{-2/3}$$

$$= \left(\frac{3}{5} E_F\right) \cdot N q$$

↳ PRESSURE OF e^- ON WALLS

E_{tot} PLAY ROLE OF INTERNAL ENERGY U
(cf. ORDINARY GAS)

IF V INCREASES U DECREASE

$$dU = -P\,dV$$

P ↑ PRESSURE OF WALLS.

$$dE_{tot} = -\frac{2}{3}\frac{E_{tot}}{V}\,dV$$

⇓

PRESSURE $P = \frac{2}{3}\frac{E_{tot}}{V}$

$$\boxed{P = \frac{2}{3}\frac{\hbar^2 k_F^5}{10\pi^2 m} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}}$$

↓

'DEGENERACY PRESSURE' (QM: $\sim \hbar^2$)

DUE TO EXCLUSION PRINCIPLE

(PREVENTS SOLIDS FROM COLLAPSING)

3) SECOND QUANTIZATION

SCHRÖDINGER EQ. FOR N - PARTICLE SYSTEM WITH INTERACTIONS

- $$\boxed{H = \sum_{i=1}^{N} H_0(x_i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} V(x_i, x_j)}$$

H_0: KINETIC ENERGY + EXTERNAL FIELDS (1 BODY OPERATOR)

V: POTENTIAL ENERGY OF INTERACTION BETWEEN PARTICLES (2 BODY OPERATOR)

e.g. COULOMB POTENTIAL $\bar{r}_i \; \overset{e^-}{\bullet} \underset{|\bar r_i - \bar r_j|}{\text{———}} \overset{e^-}{\bullet} \; \bar{r}_j$

$$V(\bar r_i, \bar r_j) = + \frac{q_i \, q_j}{4\pi \varepsilon_0 |\bar r_i - \bar r_j|}$$

- WAVE FUNCTION FOR N - PARTICLE SYSTEM

$$\Psi(x_1, \dots x_N ; t)$$

$$H \, \Psi(x_1 \dots x_N ; t) = i\hbar \frac{\partial}{\partial t} \Psi(x_1 \dots x_N ; t)$$

- EXPANSION IN COMPLETE SET OF SINGLE PARTICLE W.F.

$$H_0(i) \, \psi_{\alpha_i}(x_i) = E_i \, \psi_{\alpha_i}(x_i)$$

α_i : QUANTUM NUMBER FOR PARTICLE i (e.g. $\alpha_i = 1, 2, 3 \dots$)

e.g. HOMOGENEOUS SYSTEM

$\psi_{\alpha_i}(x_i) \rightarrow$ PLANE WAVES IN LARGE BOX WITH PERIODIC BOUNDARY CONDITIONS

$$\boxed{\Psi(x_1 \dots x_N; t) = \sum_{\underbrace{\alpha_1 \dots \alpha_N}_{\text{QUANTUM NUMBERS}}} C(\alpha_1 \dots \alpha_N; t)\, \varphi_{\alpha_1}(x_1) \cdots \varphi_{\alpha_N}(x_N)}$$

NORMALIZATION $\longrightarrow \sum_{\alpha_1 \dots \alpha_N} |C(\alpha_1 \dots \alpha_N; t)|^2 = 1$

- SCHRÖDINGER EQ.

$$i\hbar \frac{\partial}{\partial t} \sum_{\alpha_1' \dots \alpha_N'} C(\alpha_1' \dots \alpha_N'; t)\, \varphi_{\alpha_1'}(x_1) \cdots \varphi_{\alpha_N'}(x_N)$$

$$= \sum_{\alpha_1' \dots \alpha_N'} C(\alpha_1' \dots \alpha_N'; t) \sum_{k=1}^{N} H_0(x_k)\, \varphi_{\alpha_1'}(x_1) \cdots \varphi_{\alpha_N'}(x_N)$$

$$+ \sum_{\alpha_1' \dots \alpha_N'} C(\alpha_1' \dots \alpha_N'; t)\, \frac{1}{2} \sum_{k,\ell=1}^{N} V(x_k, x_\ell)\, \varphi_{\alpha_1'}(x_1) \cdots \varphi_{\alpha_N'}(x_N)$$

$\Downarrow$ MULTIPLY ON LEFT WITH

$$\int dx_1 \dots \int dx_N\, \varphi^*_{\alpha_1}(x_1) \cdots \varphi^*_{\alpha_N}(x_N)$$

$$i\hbar \frac{\partial}{\partial t} C(\alpha_1 \dots \alpha_N; t)$$

$$= \sum_{k=1}^{N} \sum_{\alpha_k'} C(\alpha_1 \dots \alpha_{k-1}\, \underline{\underline{\alpha_k'}}\, \alpha_{k+1} \dots \alpha_N, t) \cdot \int dx_k\, \varphi^*_{\alpha_k}(x_k)\, H_0(x_k)\, \varphi_{\alpha_k'}(x_k)$$

$$+ \frac{1}{2} \sum_{k,\ell} \sum_{\alpha_k'} \sum_{\alpha_\ell'} C(\alpha_1 \dots \alpha_{k-1}\, \underline{\underline{\alpha_k'}}\, \alpha_{k+1} \dots \alpha_{\ell-1}\, \underline{\underline{\alpha_\ell'}}\, \alpha_{\ell+1} \dots \alpha_N, t)$$
$$\int dx_k \int dx_\ell\, \varphi^*_{\alpha_k}(x_k)\, \varphi^*_{\alpha_\ell}(x_\ell)\, V(x_k, x_\ell)\, \varphi_{\alpha_k'}(x_k)\, \varphi_{\alpha_\ell'}(x_\ell)$$

$\Downarrow$

$\forall$ SET OF QUANTUM NUMBERS $\alpha_1 \ldots \alpha_N$

DIFF. EQ. FOR COEFFICIENTS $C(\alpha_1 \ldots \alpha_N ; t)$

$\Rightarrow$ COUPLED SET OF DIFFERENTIAL EQUATIONS

COMPLICATED !

- SYMMETRY CONDITION : IDENTICAL PARTICLES

SYMMETRY OF MANY-PARTICLE W.F. UNDER $i \longleftrightarrow j$

$$\Psi(\ldots x_i \ldots x_j \ldots ; t) = \pm \Psi(\ldots x_j \ldots x_i \ldots ; t)$$

$+ \Rightarrow$ SYMMETRIC (BOSONS)

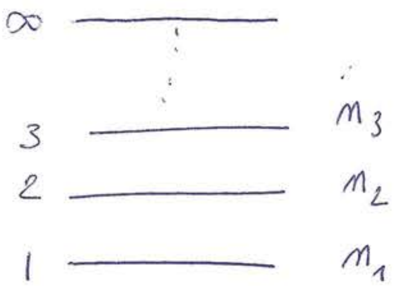
$- \Rightarrow$ ANTI-SYMMETRIC (FERMIONS)

$\Downarrow$

$$C(\ldots \alpha_i \ldots \alpha_j \ldots ; t) = \pm C(\ldots \alpha_j \ldots \alpha_i \ldots ; t)$$

- BOSONS

EVERY SINGLE PARTICLE ENERGY LEVEL CAN BE MULTIPLE OCCUPIED $\alpha_i = 1, 2, 3, \ldots$



n_i : OCCUPATION NUMBER OF STATE i

FOR BOSONS $n_i = 0, 1, 2, 3, \ldots$

$$\sum_i n_i = N$$

$$C(1\,2\,1\,2\,3\,4\,1\,2\ldots;t)$$
$$= C(\underbrace{1\ldots 1}_{n_1},\underbrace{2,\ldots 2}_{n_2},\ldots;t)$$

↳ NORMALIZATION CONDITION

$$1 = \sum_{\alpha_1\ldots\alpha_N} |C(\alpha_1\ldots\alpha_N;t)|^2$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\sum_{n_\infty=0}^{\infty} |\tilde{C}(n_1\ldots n_\infty;t)|^2 \cdot \underbrace{\frac{N!}{n_1!\ldots n_\infty!}}$$

\# REARRANGEMENTS OF N PARTICLES SUCH THAT
n_1 OCCUPY STATE 1
n_2 " " 2
⋮
n_∞ " " ∞

DEFINE $$f(n_1\ldots n_\infty;t) \equiv \tilde{C}(n_1\ldots n_\infty;t)\left(\frac{N!}{n_1!\,n_2!\ldots n_\infty!}\right)^{1/2}$$

NORMALIZATION $$\boxed{\sum_{n_1\ldots n_\infty} \underbrace{|f(n_1\ldots n_\infty;t)|^2} = 1}$$

PROBABILITY TO HAVE
n_1 PARTICLES IN STATE 1
n_2 " " " 2
n_∞ " " " ∞

$$\sum_{i=1}^{\infty} n_i = N$$

$$\hookrightarrow \Psi(x_1 \dots x_N; t) = \sum_{\alpha_1 \dots \alpha_N} C(\alpha_1 \dots \alpha_N; t)\, \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_N}(x_N)$$

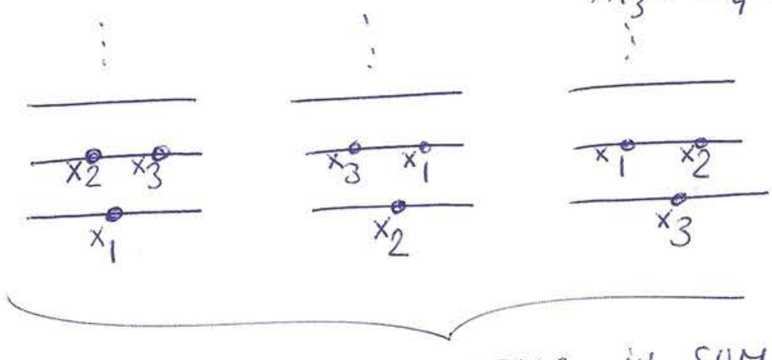
$$= \sum_{\alpha_1 \dots \alpha_N} \tilde{C}(n_1 \dots n_\infty; t)\, \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_N}(x_N)$$

$$= \sum_{n_1 \dots n_\infty} f(n_1 \dots n_\infty; t) \cdot \left(\frac{n_1! \dots n_\infty!}{N!} \right)^{1/2}$$

$$\underbrace{\sum_{\substack{\alpha_1 \dots \alpha_N \\ (n_1 \dots n_\infty)}}}_{\text{SUM OVER ALL QUANTUM NUMBERS GIVEN A SET OF OCCUPATION NUMBERS}} \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_N}(x_N)$$

e.g. $N = 3$ $\quad n_1 = 1, \quad n_2 = 2,$

$n_3 = n_4 = \dots = n_\infty = 0$



3 TERMS IN SUM

$$\frac{N!}{n_1! \dots n_\infty!}$$

$$\boxed{\Phi_{n_1 \dots n_\infty}(x_1 \dots x_N) \equiv \left(\frac{n_1! \dots n_\infty!}{N!} \right)^{1/2} \sum_{\substack{\alpha_1 \dots \alpha_N \\ (n_1 \dots n_\infty)}} \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_N}(x_N)}$$

FOR $N = 3$ $\quad n_1 = 1, \; n_2 = 2$

$n_3 = n_4 = \dots = n_\infty = 0$

$$\Phi_{1200\dots0}(x_1, x_2, x_3)$$

$$= \frac{1}{\sqrt{3}} \Big\{ \psi_1(x_1)\,\psi_2(x_2)\,\psi_2(x_3) + \psi_1(x_2)\,\psi_2(x_3)\,\psi_2(x_1) + \psi_1(x_3)\,\psi_2(x_1)\,\psi_2(x_2) \Big\}.$$

NORMALIZATION

$$\int dx_1 \dots \int dx_N \, \left| \Phi_{n_1 \dots n_\infty}(x_1 \dots x_N) \right|^2 = 1$$

$$\boxed{\Psi(x_1 \dots x_N; t) = \sum_{n_1 \dots n_\infty} f(n_1 \dots n_\infty; t)\, \Phi_{n_1 \dots n_\infty}(x_1 \dots x_N)}$$

SUM OVER OCCUPATION NUMBERS

COORDINATE SPACE WAVEFUNCTION FOR N BOSON STATE

OCCUPATION NUMBER REPRESENTATION (BOSONS)

↳ INTRODUCE TIME-INDEPENDENT STATE VECTOR

$$|n_1 \dots n_\infty\rangle$$

STATE WITH n_1 PARTICLES IN STATE 1
n_2 " " " 2
n_∞ " " " ∞

↳ ORTHONORMALITY

$$\langle n_1' \dots n_\infty' | n_1 \dots n_\infty \rangle = \delta_{n_1' n_1} \cdots \delta_{n_\infty' n_\infty}$$

↳ COMPLETENESS

$$\sum_{n_1 \dots n_\infty} |n_1 \dots n_\infty\rangle\langle n_1 \dots n_\infty| = \mathbb{1}$$

↳ CORRESPONDENCE

WAVE FUNCTION ⟺ STATE VECTOR

$$\Psi(x_1 \dots x_N; t) \iff |\Psi(t)\rangle$$

$$\Phi_{n_1 \dots n_\infty}(x_1 \dots x_N) \iff |n_1 \dots n_\infty\rangle$$

$$\boxed{|\Psi(t)\rangle = \sum_{n_1 \dots n_\infty} f(n_1 \dots n_\infty; t)\, |n_1 \dots n_\infty\rangle}$$

↳ CREATION & ANNIHILATION OPERATORS

c_i^+ (creation), c_i (annihilation) (TIME-INDEPENDENT)

IDENTICAL TO HARMONIC OSCILLATOR

DEFINE

$$[c_i, c_j^+] = \delta_{ij} \qquad [A, B] \equiv AB - BA$$

$$[c_i, c_j] = 0$$

$$[c_i^+, c_j^+] = 0.$$

$$\Downarrow$$

$$c_i^+ c_i \, |m_i\rangle = m_i \, |m_i\rangle \qquad m_i = 0, 1, 2, \ldots$$

$$c_i \, |m_i\rangle = \sqrt{m_i} \, |m_i - 1\rangle$$

$$c_i^+ \, |m_i\rangle = \sqrt{m_i + 1} \, |m_i + 1\rangle$$

$c_i^+ c_i$: HERMITIAN OPERATOR

↓

REAL EIGENVALUES m_i

↓

NUMBER OPERATOR.

↳ $$\hat{H} \, |\Psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle$$

$$= \sum_{m_1 \ldots m_\infty} i\hbar \frac{\partial f(m_1 \ldots m_\infty, t)}{\partial t} \, |m_1 \ldots m_\infty\rangle$$

↳ H_0 TERM

$$i\hbar \frac{\partial}{\partial t} C(\alpha_1 \dots \alpha_N ; t)$$

$$= \sum_{k=1}^{N} \sum_{\alpha_k'} C(\alpha_1 \dots \alpha_{k-1}\, \alpha_k'\, \alpha_{k+1} \dots \alpha_N ; t) \cdot \langle \alpha_k | H_0 | \alpha_k' \rangle$$

$$= \sum_{k=1}^{N} \sum_{\alpha_k'} \tilde{C}(n_1 \dots (n_{\alpha_k} - 1) \dots (n_{\alpha_k'} + 1) \dots n_\infty ; t) \cdot \langle \alpha_k | H_0 | \alpha_k' \rangle$$

⇓ REPLACE SUM OVER PARTICLES BY SUM OVER STATES.

$$= \sum_{\alpha_k} \sum_{\alpha_k'} n_{\alpha_k} \tilde{C}(n_1 \dots (n_{\alpha_k} - 1) \dots (n_{\alpha_k'} + 1) \dots n_\infty ; t) \langle \alpha_k | H_0 | \alpha_k' \rangle$$

↓ SIMPLIFY NOTATION

$$= \sum_i \sum_j n_i \left(\frac{n_1! \dots (n_i - 1)! \dots (n_j + 1)! \dots n_\infty!}{N!} \right)^{1/2} \cdot f(n_1 \dots n_i - 1 \dots n_j + 1 \dots n_\infty , t) \langle i | H_0 | j \rangle$$

$$= \sum_i n_i \left(\frac{n_1! \dots n_i! \dots n_\infty!}{N!} \right)^{1/2} f(n_1 \dots n_i \dots n_\infty; t) \langle i | H_0 | i \rangle$$

$$+ \sum_{i \neq j} n_i \left(\frac{n_1! \dots (n_i - 1)! \dots (n_j + 1)! \dots n_\infty!}{N!} \right)^{1/2}$$

$$\cdot f(n_1 \dots n_i - 1 \dots n_j + 1 \dots n_\infty; t) \langle i | H_0 | j \rangle$$

$$\therefore \quad i\hbar \frac{\partial}{\partial t} f(n_1 \dots n_\infty, t)$$

$$= \sum_i n_i f(n_1 \dots n_i \dots n_\infty; t) \langle i | H_0 | i \rangle$$

$$+ \sum_{i \neq j} n_i^{1/2} (n_j + 1)^{1/2} f(n_1 \dots n_i - 1 \dots n_j + 1 \dots n_\infty; t) \langle i | H_0 | j \rangle$$

$$\Downarrow$$

$$\hat{H}_0 | \Psi \rangle = \sum_{n_1 \dots n_\infty} i\hbar \frac{\partial}{\partial t} f(n_1 \dots n_\infty; t) \, | n_1 \dots n_\infty \rangle$$

$$= \sum_{n_1 \dots n_\infty} \Big\{ \sum_i n_i f(n_1 \dots n_i \dots n_\infty; t) \langle i | H_0 | i \rangle$$

$$+ \sum_{i \neq j} n_i^{1/2} (n_j + 1)^{1/2} f(n_1 \dots n_i - 1 \dots n_j + 1 \dots n_\infty; t) \langle i | H_0 | j \rangle \Big\}$$

$$| n_1 \dots n_\infty \rangle$$

IN 2° TERM

$$n_i' = n_i - 1$$

$$\Downarrow \quad n_j' = n_j + 1$$

IDENTIFY EACH TERM IN SUM $\sum_{n_1 \ldots n_\infty}$ (DROP ' IN 2° TERM)

$$\hat{H}_0 \,|\, n_1 \ldots n_\infty \rangle$$

$$= \Big\{ \sum_i n_i \, \langle i \,|\, H_0 \,|\, i \rangle \, |\, n_1 \ldots n_\infty \rangle$$

$$+ \sum_{i \neq j} (n_i + 1)^{1/2} \, (n_j)^{1/2} \, \langle i \,|\, H_0 \,|\, j \rangle \, |\, \ldots n_i + 1 \ldots n_j - 1 \ldots \rangle \Big\}$$

$$= \Big\{ \sum_i c_i^+ c_i \, \langle i \,|\, H_0 \,|\, i \rangle$$

$$+ \sum_{i \neq j} c_i^+ c_j \, \langle i \,|\, H_0 \,|\, j \rangle \Big\} \, |\, n_1 \ldots n_\infty \rangle$$

$$\therefore \quad \hat{H}_0 \,|\, n_1 \ldots n_\infty \rangle$$

$$= \sum_{ij} \langle i \,|\, H_0 \,|\, j \rangle \, c_i^+ c_j \, |\, n_1 \ldots n_\infty \rangle$$

$$\boxed{\hat{H}_0 = \sum_{i,j} \langle i \,|\, H_0 \,|\, j \rangle \, c_i^+ c_j}$$

HAMILTONIAN H_0 IN SECOND QUANTIZATION

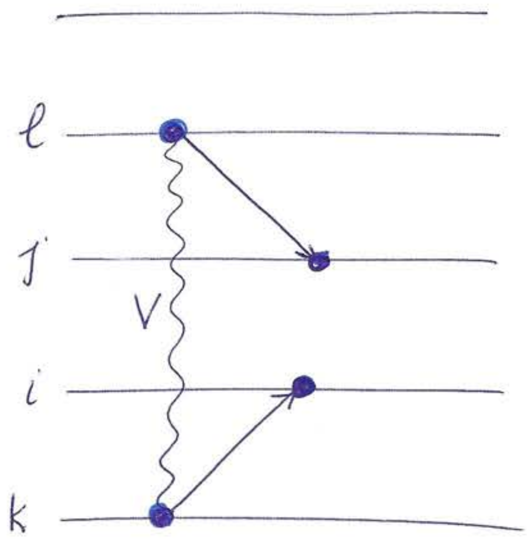
↳ V TERM

CAN BE DONE IN AN ANALOGOUS WAY

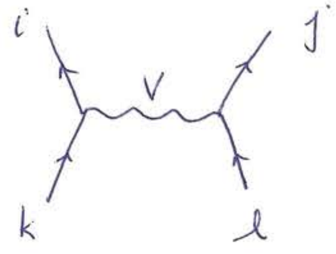
$$\therefore \quad \boxed{\begin{aligned} \hat{H} &= \sum_{i,j} \langle i | H_0 | j \rangle c_i^{+} c_j \\ &+ \frac{1}{2} \sum_{ijkl} \langle ij | V | kl \rangle c_i^{+} c_j^{+} c_l c_k \end{aligned}}$$

TOTAL HAMILTONIAN IN 2° QUANTIZATION

INTERPRETATION INTERACTION TERM



OR DIAGRAMMATIC



DUE TO V : 2 PARTICLES ANNIHILATED IN STATE k, l

& CREATED IN STATE i, j

OCCUPATION NUMBER REPRESENTATION (FERMIONS)

$\hookrightarrow \quad C(\ldots \alpha_i \ldots \alpha_j \ldots; t) = - \, C(\ldots \alpha_j \ldots \alpha_i \ldots; t)$.

$\forall i: \; n_i = 0, 1$ (PAULI PRINCIPLE)

$$\hookrightarrow \quad \Psi(x_1 \ldots x_N; t) = \sum_{n_1 \ldots n_\infty = 0}^{1} f(n_1 \ldots n_\infty; t) \underbrace{\Phi_{n_1 \ldots n_\infty}(x_1 \ldots x_N)}_{\Downarrow \text{ TOTALLY ANTI-SYMMETRIC}}$$

$$\Phi_{n_1 \ldots n_\infty}(x_1 \ldots x_N) \equiv \left(\frac{n_1! \ldots n_\infty!}{N!} \right)^{1/2} \underbrace{\begin{vmatrix} \varphi_{\alpha_1}(x_1) & \ldots & \varphi_{\alpha_1}(x_N) \\ \vdots & & \\ \varphi_{\alpha_N}(x_1) & \ldots & \varphi_{\alpha_N}(x_N) \end{vmatrix}}_{\text{SLATER DETERMINANT}}$$

$\alpha_1 \ldots \alpha_N$ QUANTUM NUMBERS OF N OCCUPIED STATES ORDERED SUCH THAT $\alpha_1 < \alpha_2 < \ldots < \alpha_N$.

$\Rightarrow$ e.g. $N = 2$.

$$\Phi_{110\ldots0}(x_1, x_2) = \frac{1}{\sqrt{2}} \left\{ \varphi_1(x_1) \varphi_2(x_2) - \varphi_1(x_2) \varphi_2(x_1) \right\}$$

$\Rightarrow$ e.g. $N = 3$

$$\begin{aligned} \Phi_{1110\ldots0}(x_1, x_2, x_3) = \frac{1}{\sqrt{6}} \Big\{ & \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) - \varphi_1(x_1)\varphi_2(x_3)\varphi_3(x_2) \\ & + \varphi_1(x_2)\varphi_2(x_3)\varphi_3(x_1) - \varphi_1(x_2)\varphi_2(x_1)\varphi_3(x_3) \\ & + \varphi_1(x_3)\varphi_2(x_1)\varphi_3(x_2) - \varphi_1(x_3)\varphi_2(x_2)\varphi_3(x_1) \Big\} \end{aligned}$$

$$\boxed{|\Psi(t)\rangle = \sum_{n_1 \dots n_\infty = 0}^{1} f(n_1 \dots n_\infty ; t) \, | n_1 \dots n_\infty \rangle}$$

ANTI - COMMUTATORS. $n_i = 0, 1$

FOR FERMIONS.

$$\{A, B\} \equiv [A, B]_+ \equiv AB + BA$$

ANTI- COMMUTATION RULES FOR CREATION / ANNIHILATION OPERATORS

DEFINE

$$\{c_i, c_j^+\} = \delta_{ij}$$

$$\{c_i, c_j\} = 0$$

$$\{c_i^+, c_j^+\} = 0$$

$\Downarrow$

$\rightsquigarrow$ IF $i = j$

$$\{c_i, c_i\} = 0 \implies c_i^2 = 0$$

$$\{c_i^+, c_i^+\} = 0 \implies c_i^{+2} = 0$$

$$(c_i^+)^2 |0\rangle = 0$$

ONE CANNOT CREATE 2 PARTICLES IN SAME QUANTUM STATE (PAULI PRINCIPLE).

⇝ NUMBER OPERATOR $N_i \equiv c_i^+ c_i$

$$\{c_i, c_i^+\} = 1 \quad \Rightarrow \quad c_i^+ c_i = 1 - c_i c_i^+$$

$$\Downarrow$$

$$(c_i^+ c_i)^2 = (1 - c_i c_i^+)(1 - c_i c_i^+)$$

$$= 1 - 2 c_i c_i^+ + c_i c_i^+ c_i c_i^+$$

$$= 1 - 2 c_i c_i^+ + c_i (1 - c_i \underbrace{\cancel{c_i^+}) c_i^+}_{0}$$

$$= 1 - c_i c_i^+$$

$$= c_i^+ c_i$$

$\therefore \quad \underline{N_i^2 = N_i} \quad \Rightarrow \quad$ EIGENVALUES $n_i = 0, 1$

$$N_i \, |n_i\rangle = n_i \, |n_i\rangle$$

⇝ $c_i^+ |\underset{\substack{\uparrow \\ \text{STATE } i}}{0}\rangle = |\underset{\substack{\uparrow \\ \text{STATE } i}}{1}\rangle \qquad c_i |1\rangle = |0\rangle$

$$c_i^+ |1\rangle = 0 \qquad c_i |0\rangle = 0$$

⇝ ANTI-COMMUTATORS : IMPORTANT TO KEEP TRACK OF SIGNS !

DEFINE

$$\boxed{|n_1 \ldots n_\infty\rangle = (c_1^+)^{n_1} (c_2^+)^{n_2} \ldots (c_\infty^+)^{n_\infty} |0\rangle}$$

e.g. ANNIHILATE PARTICLE IN STATE s $(n_s = 1)$

$$c_s \,|n_1 \dots n_s \dots n_\infty\rangle = \underbrace{(-1)^{S_s}}_{\Downarrow} (c_1^+)^{n_1} \dots \underbrace{c_s (c_s^+)}_{1 - c_s^+ c_s} \dots (c_\infty^+)^{n_\infty} |0\rangle$$

PHASE FACTOR

$$S_s = n_1 + \dots + n_{s-1}$$

e.g. CREATE PARTICLE IN STATE s $(n_s = 0)$

$$c_s^+ \,|n_1 \dots 0 \dots n_\infty\rangle = (-1)^{S_s} (c_1^+)^{n_1} \dots c_s^+ \dots (c_\infty^+)^{n_\infty} |0\rangle$$

$\therefore$

$$c_s \,|\dots n_s \dots\rangle = \begin{cases} (-1)^{S_s} \sqrt{n_s} \,|\dots n_s - 1 \dots\rangle, & \text{IF } n_s = 1 \\ 0, & \text{IF } n_s = 0 \end{cases}$$

$$c_s^+ \,|\dots n_s \dots\rangle = \begin{cases} (-1)^{S_s} \sqrt{n_s + 1} \,|\dots n_s + 1 \dots\rangle, & \text{IF } n_s = 0 \\ 0, & \text{IF } n_s = 1 \end{cases}$$

$$c_s^+ c_s \,|\dots n_s \dots\rangle = n_s \,|\dots n_s \dots\rangle$$

SAME FORM AS FOR BOSONS

EXCEPT FOR PHASE FACTOR !

↳ HAMILTONIAN (FERMIONS) IN SECOND QUANTIZED FORM

$$\boxed{\begin{aligned}\hat{H} &= \sum_{i,j} \langle i | H_0 | j \rangle \, c_i^+ c_j \\ &+ \frac{1}{2} \sum_{ijkl} \langle ij | V | kl \rangle \, c_i^+ c_j^+ c_l c_k\end{aligned}}$$

⇑

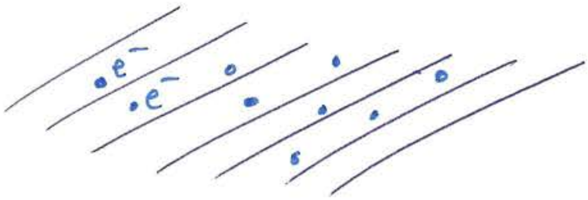
ORDERING IMPORTANT !

CHECK ORDERING

HOMOGENEOUS ELECTRON GAS (JELLIUM MODEL)

↳ FIRST APPROXIMATION TO A METAL OR A PLASMA

∴ INTERACTING e^- GAS PLACED IN A UNIFORMLY DISTRIBUTED POSITIVE BACKGROUND (IONS)



TOTAL SYSTEM IS ELECTRICAL NEUTRAL.

↳ SYSTEM IN LARGE CUBICAL BOX OF LENGTH L (AT END OF CALCULATION $L \to \infty$). $V = L^3$.

e^- WAVE FUNCTION $\psi_{\bar{k} s}(\bar{x}) = \frac{1}{\sqrt{V}} e^{i \bar{k} \cdot \bar{x}} \eta_s$

↑ SPIN PROJ $s = \pm \frac{1}{2}$

$$\eta_{s=+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \eta_{s=-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

PERIODIC BOUNDARY CONDITION $k_i = \frac{2\pi}{L} n_i$

$i = x, y, z$

$n_i = 0, \pm 1, \pm 2, \ldots$

↳ TOTAL : N ELECTRONS IN BOX OF VOLUME V

AT END $\left.\begin{matrix} N \to \infty \\ V \to \infty \end{matrix}\right\} n \equiv \frac{N}{V}$ STAYS CONSTANT

HAMILTONIAN

$$H = \underset{\text{ELECTRON}}{H_{el}} + \underset{\text{BACKGROUND}}{H_b} + \underset{e^- \text{ BACKGROUND INTERACTION}}{H_{el-b}}$$

$$H_{el} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \underbrace{\frac{1}{2} e^2 \sum_{i \neq j} \frac{e^{-\mu |\bar{r}_i - \bar{r}_j|}}{|\bar{r}_i - \bar{r}_j|}}_{e^- \text{ REPULSION}}$$

$e^{-\mu|\,|}$ INTRODUCED FOR CONVERGENCE

AT END : $\mu \to 0$

$$H_b = \frac{1}{2} e^2 \int d^3\bar{x} \int d^3\bar{x}' \; \underbrace{n(\bar{x})}_{\text{DENSITY OF POSITIVELY CHARGED BACKGROUND PARTICLES.}} n(\bar{x}') \frac{e^{-\mu |\bar{x} - \bar{x}'|}}{|\bar{x} - \bar{x}'|}$$

$$H_{el-b} = \underset{\text{ATTRACTIVE INTERACTION BETWEEN } e^- \text{ AND BACKGROUND}}{-} e^2 \sum_{i=1}^{N} \int d^3\bar{x} \; n(\bar{x}) \frac{e^{-\mu |\bar{x} - \bar{r}_i|}}{|\bar{x} - \bar{r}_i|}$$

$\rightsquigarrow$ UNIFORM BACKGROUND $n(\bar{x}) = \frac{N}{V}$

$$H_b = \frac{1}{2} e^2 \left(\frac{N}{V}\right)^2 \int d^3\bar{x} \int d^3\bar{x}' \, \frac{e^{-\mu|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|}$$

$$= \frac{1}{2} e^2 \left(\frac{N}{V}\right)^2 \cdot V \int d^3\bar{r} \, \frac{e^{-\mu|\bar{r}|}}{|\bar{r}|}$$

$$= \frac{1}{2} e^2 \left(\frac{N}{V}\right)^2 \cdot V \cdot 4\pi \underbrace{\int_0^\infty dr \, r \, e^{-\mu r}}_{\frac{1}{\mu^2}} \quad (\text{LIMIT } L \to \infty)$$

$$= \frac{1}{2} e^2 \frac{N^2}{V} \cdot \frac{4\pi}{\mu^2}$$

$\hookrightarrow$ DIVERGES IN LIMIT $\mu \to 0$.

$$H_{el-b} = - e^2 \frac{N}{V} \sum_{i=1}^{N} \int d^3\bar{x} \, \frac{e^{-\mu|\bar{x}-\bar{r}_i|}}{|\bar{x}-\bar{r}_i|}$$

$\downarrow$ TRANSLATIONAL INVARIANCE

$$= - e^2 \frac{N^2}{V} \int d^3\bar{r} \, \frac{e^{-\mu|\bar{r}|}}{|\bar{r}|}$$

$$= - e^2 \frac{N^2}{V} \cdot \frac{4\pi}{\mu^2}$$

$$\therefore \quad \underline{\underline{H = -\frac{1}{2} e^2 \frac{N^2}{V} \cdot \frac{4\pi}{\mu^2} + H_{el}}}$$

$\rightsquigarrow$ $\hat{H}_{el}$ IN SECOND QUANTIZATION

e^- STATES CHARACTERIZED BY $i = \bar{k}, s$

- KINETIC ENERGY

$$\hat{T} = \sum_{i,j} \langle i | T | j \rangle \, c_i^+ c_j$$

$$\Downarrow$$

$$\hat{T} = \sum_{\bar{k} s} \sum_{\bar{k}' s'} \langle \bar{k} s | T | \bar{k}' s' \rangle \, c^+_{\bar{k} s} \, c_{\bar{k}' s'}$$

$$\langle \bar{k} s | T | \bar{k}' s' \rangle$$

$$= \int d^3\bar{x} \; \psi^*_{\bar{k}s}(\bar{x}) \left(- \frac{\hbar^2 \nabla^2}{2m} \right) \psi_{\bar{k}'s'}(\bar{x})$$

$$= \frac{+\hbar^2}{2mV} \underbrace{\eta_s^+ \eta_{s'}}_{\delta_{ss'}} \cdot \int d^3\bar{x} \; e^{-i\bar{k}.\bar{x}} \, \bar{k}'^2 \, e^{i\bar{k}'.\bar{x}}$$

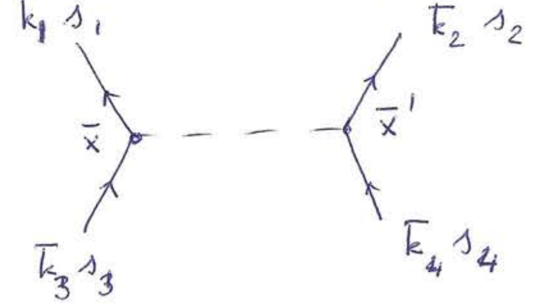
$$= \frac{\hbar^2 \bar{k}'^2}{2mV} \, \delta_{ss'} \cdot \underbrace{\int d^3\bar{x} \; e^{i(\bar{k}' - \bar{k}).\bar{x}}}_{V \cdot \delta_{\bar{k}\bar{k}'}}$$

$$\therefore \quad \boxed{\hat{T} = \sum_{\bar{k} s} \frac{\hbar^2 \bar{k}^2}{2m} \, c^+_{\bar{k}s} \, c_{\bar{k}s}}$$

- POTENTIAL ENERGY

$$\hat{V} = \frac{1}{2} \sum_{\bar{k}_1 s_1} \sum_{\bar{k}_2 s_2} \sum_{\bar{k}_3 s_3} \sum_{\bar{k}_4 s_4} \langle \bar{k}_1 s_1, \bar{k}_2 s_2 | V | \bar{k}_3 s_3, \bar{k}_4 s_4 \rangle \cdot c^+_{\bar{k}_1 s_1} c^+_{\bar{k}_2 s_2} c_{\bar{k}_4 s_4} c_{\bar{k}_3 s_3}$$

GRAPHICALLY



$$\langle \bar{k}_1 s_1, \bar{k}_2 s_2 | V | \bar{k}_3 s_3, \bar{k}_4 s_4 \rangle$$

$$= \frac{1}{V^2} \int d^3\bar{x} \int d^3\bar{x}' \, e^{-i(\bar{k}_1 - \bar{k}_3)\cdot\bar{x}} \, e^{-i(\bar{k}_2 - \bar{k}_4)\cdot\bar{x}'}$$

$$\cdot \, e^2 \frac{e^{-\mu|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} \cdot \underbrace{\eta^+_{s_1} \eta^+_{s_2} \eta_{s_3} \eta_{s_4}}_{\delta_{s_1 s_3} \, \delta_{s_2 s_4}}$$

(SPIN DOES NOT CHANGE)

↳ INTERACTION IS SPIN-INDEPENDENT

$\downarrow$ $\bar{y} \equiv \bar{x} - \bar{x}'$

ELIMINATE $\bar{x}$

$$= \frac{1}{V^2} \cdot e^2 \delta_{s_1 s_3} \delta_{s_2 s_4} \int d^3\bar{x}' \, e^{-i(\bar{k}_1 + \bar{k}_2 - \bar{k}_3 - \bar{k}_4)\cdot\bar{x}'}$$

$$\cdot \int d^3\bar{y} \, \frac{e^{-\mu|\bar{y}|}}{|\bar{y}|} \, e^{-i(\bar{k}_1 - \bar{k}_3)\cdot\bar{y}}$$

$$= \frac{1}{V^2}\, e^2\, \delta_{s_1 s_3}\, \delta_{s_2 s_4} \cdot V \underbrace{\delta_{\bar{k}_1+\bar{k}_2,\, \bar{k}_3+\bar{k}_4}}_{\substack{\text{MOMENTUM}\\ \text{CONSERVATION}\\ \text{(UNIFORM SYSTEM)}}}$$

$$\cdot \underbrace{\int d^3\bar{y}\ \frac{e^{-\mu y}}{y}\ e^{-i(\bar{k}_1-\bar{k}_3)\cdot\bar{y}}}$$

$$2\pi \int_0^\infty dy\ y\, e^{-\mu y} \underbrace{\int_{-1}^{1} d\cos\theta\ e^{i|\bar{k}_1-\bar{k}_3| y\cos\theta}}$$

$$\underbrace{\frac{1}{-i|\bar{k}_1-\bar{k}_3|y}\left(e^{-i|\bar{k}_1-\bar{k}_3|y} - e^{+i|\bar{k}_1-\bar{k}_3|y}\right)}$$

$$\frac{2\pi}{i|\bar{k}_1-\bar{k}_3|} \int_0^\infty dy \left\{ e^{(i|\bar{k}_1-\bar{k}_3| - \mu)y} - e^{-(+i|\bar{k}_1-\bar{k}_3|+\mu)y} \right\}$$

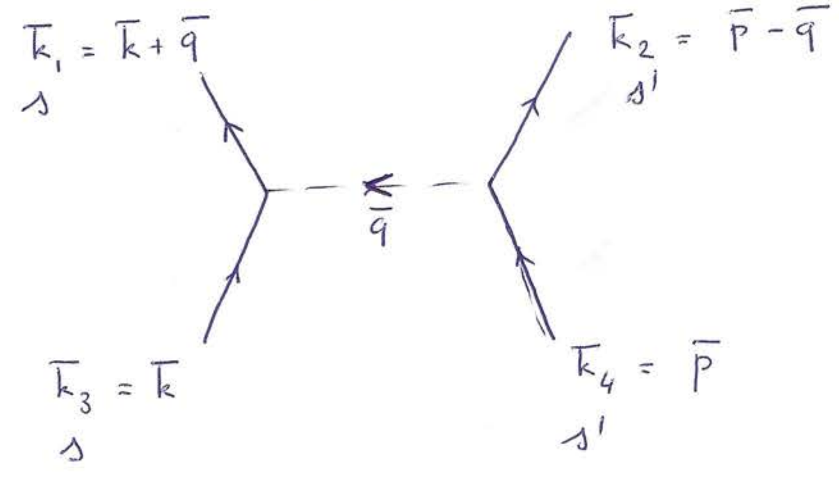
$$= \frac{2\pi}{i|\bar{k}_1-\bar{k}_3|} \cdot \underbrace{\left\{ -\frac{1}{i|\bar{k}_1-\bar{k}_3|-\mu} + \frac{1}{-i|\bar{k}_1-\bar{k}_3|-\mu} \right\}}_{\frac{2i|\bar{k}_1-\bar{k}_3|}{|\bar{k}_1-\bar{k}_3|^2+\mu^2}}$$

$$= \frac{4\pi}{|\bar{k}_1-\bar{k}_3|^2+\mu^2}$$

$$\therefore\ \boxed{\hat{V} = \frac{e^2}{2V} \sum_{\bar{k}_1}\sum_{\bar{k}_2}\sum_{\bar{k}_3}\sum_{\bar{k}_4}\sum_{s_1 s_2} \delta_{\bar{k}_1+\bar{k}_2,\, \bar{k}_3+\bar{k}_4}\ \frac{4\pi}{|\bar{k}_1-\bar{k}_3|^2+\mu^2}\ C^+_{\bar{k}_1 s_1} C^+_{\bar{k}_2 s_2} C_{\bar{k}_4 s_2} C_{\bar{k}_3 s_1}}$$

• SIMPLIFY USING MOMENTUM CONSERVATION

$$\delta_{\bar{k}_1+\bar{k}_2,\ \bar{k}_3+\bar{k}_4}$$



$$\therefore\ \hat{V} = \frac{e^2}{2V} \sum_{s,s'} \sum_{\bar{k}} \sum_{\bar{p}} \sum_{\bar{q}} \frac{4\pi}{|\bar{q}|^2+\mu^2} \cdot c^{+}_{\bar{k}+\bar{q}\, s}\, c^{+}_{\bar{p}-\bar{q}\, s'}\, c_{\bar{p} s'}\, c_{\bar{k} s}$$

$\downarrow$ DIVERGES FOR $|\bar{q}| = 0$ (WHEN $\mu \to 0$)

$\therefore$ SPLIT $\sum_{\bar{q}} \cdots = (\bar{q} = 0 \text{ TERM}) + \sum_{\bar{q}\neq 0} \cdots$

$$\hat{V} = \frac{e^2}{2V} \sum_{s,s'} \frac{4\pi}{\mu^2} \cdot \sum_{\bar{p}} \sum_{\bar{k}} c^{+}_{\bar{k} s}\, c^{+}_{\bar{p} s'}\, c_{\bar{p} s'}\, c_{\bar{k} s}$$

$+$ TERM WITH $\bar{q} \neq 0$

USE $c^{+}_{\bar{k}s}\, c^{+}_{\bar{p}s'}\, c_{\bar{p}s'}\, c_{\bar{k}s}$

$$= c^{+}_{\bar{k}s}\, c^{+}_{\bar{p}s'} \left(- c_{\bar{k}s}\, c_{\bar{p}s'} \right)$$

$$= - c^{+}_{\bar{k}s} \left(\delta_{ss'}\, \delta_{\bar{k}\bar{p}} - c_{\bar{k}s}\, c^{+}_{\bar{p}s'} \right) c_{\bar{p}s'}$$

$$= - \delta_{ss'}\, \delta_{\bar{k}\bar{p}}\, c^{+}_{\bar{k}s}\, c_{\bar{k}s} + \left(c^{+}_{\bar{k}s}\, c_{\bar{k}s} \right) \left(c^{+}_{\bar{p}s'}\, c_{\bar{p}s'} \right)$$

$$\Rightarrow \quad \hat{V} = \frac{e^2}{2V} \frac{4\pi}{\mu^2} \left(\hat{N}^2 - \hat{N} \right)$$

$$+ \text{ TERM WITH } \bar{q} \neq 0$$

WITH PARTICLE NUMBER OPERATOR $\hat{N} = \sum_{\bar{k}s} c^{+}_{\bar{k}s}\, c_{\bar{k}s}$

$\Downarrow$

FOR SYSTEMS WITH FIXED N

$\hat{N}^2 - \hat{N} \curvearrowright$ EIGENVALUE $N(N-1)$

FIRST TERM $\underbrace{\frac{e^2}{2V} N^2 \frac{4\pi}{\mu^2}}_{\text{CANCELS } H_b + H_{el-b}} - \underbrace{\frac{e^2}{2V} N \frac{4\pi}{\mu^2}}_{\text{ENERGY PER } e^-}$

CANCELS $H_b + H_{el-b}$ ENERGY PER e^- $\sim \frac{1}{V} \xrightarrow[V \to \infty]{} 0$

⇝ TOTAL ENERGY $\hat{H} = \hat{H}_{el} + \hat{H}_b + \hat{H}_{el-b}$

$$\boxed{\hat{H} = \sum_{\bar{k}s} \frac{\hbar^2 \bar{k}^2}{2m} c^+_{\bar{k}s} c_{\bar{k}s} + \frac{e^2}{2V} \sum_{\bar{k}} \sum_{\bar{p}} \sum_{\bar{q}\neq 0} \sum_{ss'} \frac{4\pi}{\bar{q}^2} c^+_{\bar{k}+\bar{q}\,s} c^+_{\bar{p}-\bar{q}\,s'} c_{\bar{p}s'} c_{\bar{k}s}}$$

WE TOOK LIMIT $\mu \to 0$

(FIRST $V \to \infty$, THEN $\mu \to 0$).

$\Downarrow$

INFINITE TERMS CANCEL
IN THERMODYNAMIC LIMIT
(ELECTRIC NEUTRALITY).

⇝ DIMENSIONLESS VARIABLES

r_0 : INTERPARTICLE SPACING $\quad \frac{V}{N} \equiv \frac{4}{3}\pi r_0^3$

a_0 : "BOHR RADIUS" $\quad a_0 \equiv \frac{\hbar^2}{me^2}$

(DUE TO COULOMB INTERACTION)

r_s : DIMENSIONLESS $\quad r_s \equiv \frac{r_0}{a_0}$.

$$\tilde{V} \equiv \frac{V}{r_0^3}$$

$$\tilde{k} \equiv r_0 \bar{k} \qquad \tilde{p} \equiv r_0 \bar{p} \qquad \tilde{q} \equiv r_0 \bar{q}$$

$$\therefore \quad \boxed{\hat{H} = \frac{e^2}{a_0 r_s^2} \left\{ \sum_{\tilde{k} s} \frac{1}{2} \tilde{k}^2 c^+_{\tilde{k} s} c_{\tilde{k} s} + \frac{r_s}{2\tilde{V}} \sum_{\tilde{k}} \sum_{\tilde{p}} \sum_{\tilde{q} \neq 0} \sum_{s s'} \frac{4\pi}{\tilde{q}^2} c^+_{\tilde{k}+\tilde{q}\, s} c^+_{\tilde{p}-\tilde{q}\, s'} c_{\tilde{p} s'} c_{\tilde{k} s} \right\}}$$

IN LIMIT $r_s \to 0$ (HIGH DENSITY LIMIT)

POTENTIAL BECOMES A SMALL PERTURBATION

$\downarrow$

EXERCISE: CALCULATE GROUND STATE ENERGY OF HIGH DENSITY e^- GAS. IN 1° ORDER PERTURBATION THEORY