Exercise sheet 2
Theoretical Physics 5 : SS 2023

24.04.2023

Write your name and your tutor's name on every page you hand in. Please staple said pages together.

Exercise 0.

How much time did you take to complete this homework sheet?

Exercise 1. (40 points): Boson wave-functions

Explicitly write the following N-boson wave functions in terms of single-boson wave functions $\psi_i(x_j)$

a) *(10 p.)* $\Phi_{0300\cdots 0}(x_1, x_2, x_3)$

b) *(10 p.)* $\Phi_{3010\cdots 0}(x_1, x_2, x_3, x_4)$

c) *(10 p.)* $\Phi_{2110\cdots 0}(x_1, x_2, x_3, x_4)$

You should have found that the answer to question 1a consisted of one term only. How many terms will the following wave functions contain? Do not write the wave functions explicitly!

d) *(5 p.)* $\Phi_{1220\cdots 0}(x_1, \ldots, x_5)$

e) *(5 p.)* $\Phi_{5061\cdots 0}(x_1, \ldots, x_{12})$

Exercise 2. (60 points): The second-quantized potential

The Hamiltonian for an N-particle system is given by

$$H = \sum_{k=1}^{N} H_0(x_k) + \frac{1}{2} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq k}}^{N} V(x_k, x_l).$$

In the lecture notes you have seen that in the second-quantized formalism the first term can be rewritten as

$$\hat{H}_0 = \sum_{i,j} \langle i|H_0|j\rangle \, c_i^\dagger c_j.$$

In this exercise we are going to show that the potential term can be rewritten into

$$\hat{V} = \frac{1}{2} \sum_{i,j,k,l} \langle ij|V|kl\rangle \, c_i^\dagger c_j^\dagger c_l c_k.$$

a) (*10 p.*) Starting from

$$\Psi(x_1, \ldots, x_N, t) = \sum_{\alpha_1, \ldots, \alpha_N} C(\alpha_1, \ldots, \alpha_N, t) \psi_{\alpha_1}(x_1) \ldots \psi_{\alpha_N}(x_N),$$
$$i\hbar \partial_t \Psi(x_1, \ldots, x_N, t) = H \Psi(x_1, \ldots, x_N, t)$$

derive the differential equation for the coefficients $C(\alpha_1, \ldots, \alpha_N, t)$,

$$\begin{aligned} & i\hbar \partial_t C(\alpha_1, \ldots, \alpha_N, t) \\ & = \sum_{k=1}^{N} \sum_{\alpha_W} \langle \alpha_k|H_0|\alpha_W\rangle \, C(\alpha_1, \ldots, \alpha_{k-1}, \alpha_W, \alpha_{k+1}, \ldots, \alpha_N, t) \\ & + \frac{1}{2} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq k}}^{N} \sum_{\alpha_W} \sum_{\alpha'_W} \langle \alpha_k \alpha_l|V|\alpha_W \alpha'_W\rangle \\ & \quad \times C(\alpha_1, \ldots, \alpha_{k-1}, \alpha_W, \alpha_{k+1}, \ldots, \alpha_{l-1}, \alpha'_W, \alpha_{l+1}, \ldots, \alpha_N, t). \end{aligned}$$

b) (*20 p.*) Show that one can rewrite the potential term into

$$\begin{aligned} & \frac{1}{2} \sum_i \sum_j \sum_k \sum_l n_i \, (n_j - \delta_{ij}) \, \langle ij|V|kl\rangle \\ & \quad \times \tilde{C}(n_1, \ldots, n_i - 1, \ldots, n_k + 1, \ldots, n_j - 1, \ldots, n_l + 1, \ldots, n_\infty, t), \end{aligned}$$

where $\tilde{C}$ is defined as in the lecture notes.

(Hint: if you are stuck, take a look at the suggested literature for this course.)

c) (*10 p.*) Use

$$f(n_1, \ldots, n_\infty, t) = \tilde{C}(n_1, \ldots, n_\infty, t) \left(\frac{N!}{n_1! \ldots n_\infty!} \right)^{1/2}$$

to derive a differential equation for f.

d) (*20 p.*) By acting with the potential term on the state

$$|\Psi\rangle = \sum_{n_1, \ldots, n_\infty} f(n_1, \ldots, n_\infty, t) \, |n_1 \cdots n_\infty\rangle$$

show that

$$\hat{V} = \frac{1}{2} \sum_{i,j,k,l} \langle ij|V|kl\rangle \, c_i^\dagger c_j^\dagger c_l c_k.$$