III

RELATIVISTIC BOSONS

III 1) FREE SPIN 0 PARTICLES :
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KLEIN - GORDON FIELD

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KLEIN - GORDON FIELD

1) FREE SPIN-0 PARTICLES: KLEIN-GORDON EQUATION

- RELATIVISTIC NOTATION (REVIEW)

DESCRIPTION OF PROCESSES AT HIGH-ENERGIES
↳ WHERE CREATION OF PARTICLES & ANTI-PARTICLES
⇓
REQUIRES RELATIVISTIC TREATMENT

↳ 4-VECTOR SPACE-TIME

- $x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$
  ↳ CONTRAVARIANT
- $x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z)$
  ↳ COVARIANT

- $x_\mu = g_{\mu\nu} x^\nu$ ⇝ EINSTEIN SUMMATION CONVENTION (ν IS SUMMED OVER $\nu = 0, \dots 3$)

$$= \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$$

WITH $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
↓
METRIC TENSOR

$$g^{\mu\sigma} g_{\sigma\nu} = g^\mu_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- RELATIVISTIC 'LENGTH' OF 4-VECTOR

$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$$
$$= c^2t^2 - x^2 - y^2 - z^2$$

$x_\mu x^\mu$ IS <u>INVARIANT</u> UNDER LORENTZ TF.

e.g. LORENTZ BOOST ALONG X-AXIS

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$c^2t'^2 - x'^2 - y'^2 - z'^2$$
$$= \gamma^2\left[c^2\left(t^2 - \cancel{2\frac{v}{c^2}tx} + \frac{v^2}{c^4}x^2\right) - \left(v^2t^2 - \cancel{2xvt} + x^2\right)\right] - y^2 - z^2$$
$$= \frac{1}{(1 - v^2/c^2)}\left[c^2\left(1 - \frac{v^2}{c^2}\right)t^2 - \left(1 - \frac{v^2}{c^2}\right)x^2\right] - y^2 - z^2$$
$$\stackrel{!}{=} c^2t^2 - x^2 - y^2 - z^2$$

RELATIVISTIC FOUR-MOMENTUM

- $p^{\mu} = (p^0, p^1, p^2, p^3) \equiv \left(\frac{E}{c}, p_x, p_y, p_z\right)$

  E : ENERGY

- $p_{\mu} = \left(\frac{E}{c}, -p_x, -p_y, -p_z\right)$

- $p_{\mu} p^{\mu} = \frac{E^2}{c^2} - \bar{p}^2 \qquad (\bar{p}^2 \equiv p_x^2 + p_y^2 + p_z^2)$

<u>EINSTEIN</u> : $E^2 = c^2 \bar{p}^2 + m_0^2 c^4$

WITH m_0 : REST MASS

$\Downarrow$

$$p_{\mu} p^{\mu} = m_0^2 c^2$$

FOUR-MOMENTUM OPERATOR

- $$\hat{p}^{\mu} \equiv i\hbar \nabla^{\mu} = i\hbar \frac{\partial}{\partial x_{\mu}}$$

  $$= \left(i\hbar \frac{\partial}{\partial(ct)}, -i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}\right)$$

  $$= i\hbar \left(\frac{\partial}{\partial(ct)}, -\bar{\nabla}\right)$$

- $$\hat{p}_{\mu} \equiv i\hbar \nabla_{\mu} = i\hbar \frac{\partial}{\partial x^{\mu}}$$

  $$= i\hbar \left(\frac{\partial}{\partial(ct)}, +\bar{\nabla}\right)$$

NOTATION $\nabla^{\mu} \equiv \partial^{\mu} = \frac{\partial}{\partial x_{\mu}}, \quad \nabla_{\mu} \equiv \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$

- $$\hat{p}^\mu \hat{p}_\mu = -\hbar^2 \nabla_\mu \nabla^\mu$$
$$= -\hbar^2 \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right)$$

- COMMUTATOR

$$[\hat{p}^\mu, x^\nu] = \hat{p}^\mu x^\nu - x^\nu \hat{p}^\mu$$
$$= i\hbar \left(\nabla^\mu x^\nu - x^\nu \nabla^\mu \right)$$
$$= i\hbar \left(\nabla^\mu x^\nu \right)$$
$$= i\hbar \left(\frac{\partial}{\partial x_\mu} g^{\nu\sigma} x_\sigma \right)$$
$$= i\hbar \; g^{\nu\sigma} g^\mu_\sigma$$
$$= i\hbar \; g^{\mu\nu}$$

↳ FOUR-VECTOR POTENTIAL.

- $$A^\mu = (A^0, \underset{\substack{\uparrow \\ \text{3-VECTOR}}}{\bar{A}}) = (A^0, A_x, A_y, A_z)$$

- $$A_\mu = (A^0, -\bar{A})$$

- FIELD TENSOR

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$= \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

• KLEIN - GORDON EQUATION

↳ NON - RELATIVISTIC SCHRÖDINGER EQUATION

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m_0} \nabla^2 + V \right] \Psi$$

$$\updownarrow$$

$$\hat{H} = \frac{\hat{p}^2}{2m_0} + \hat{V}$$

ENERGY FREE PARTICLE $\quad E = \frac{p^2}{2m_0}$ (NON - RELATIVISTIC KINETIC ENERGY)

ENERGY OPERATOR $\quad \hat{E} = i\hbar \frac{\partial}{\partial t}$

MOMENTUM " $\quad \hat{\bar{p}} = -i\hbar \bar{\nabla}$

$$\hat{p}^\mu = i\hbar \partial^\mu$$

↳ RELATIVISTIC KLEIN - GORDON EQUATION

- RELATIVISTIC ENERGY RELATION

$$E^2 = c^2 \bar{p}^2 + m_0^2 c^4$$

$$\downarrow$$

$$\frac{E^2}{c^2} - \bar{p}^2 = m_0^2 c^2$$

$$p_\mu p^\mu = m_0^2 c^2$$

- WAVE EQUATION

$$\hat{p}_\mu \hat{p}^\mu \Psi = m_0^2 c^2 \Psi$$

$$\downarrow$$

$$-\hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\Psi = m_0^2c^2\,\Psi$$

$$\downarrow$$

$$\boxed{\left(\partial_\mu\partial^\mu+\frac{m_0^2c^2}{\hbar^2}\right)\Psi=0}$$

$$\partial_\mu\partial^\mu \equiv \Box \qquad \text{d'ALEMBERT OPERATOR}$$

- PLANE - WAVE SOLUTION : FREE PARTICLE

$$\Psi(\bar{x},t)=A\,e^{-\frac{i}{\hbar}p_\mu x^\mu}=A\,e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{x})}$$

$$\partial_\mu\partial^\mu\Psi=-\frac{1}{\hbar^2}p_\mu p^\mu\,\Psi$$

$$\left(\partial_\mu\partial^\mu+\frac{m_0^2c^2}{\hbar^2}\right)\Psi=0$$

$$\Updownarrow$$

$$-\frac{p_\mu p^\mu}{\hbar^2}+\frac{m_0^2c^2}{\hbar^2}=0$$

$$\underline{\underline{p_\mu p^\mu = m_0^2c^2}} \quad \checkmark$$

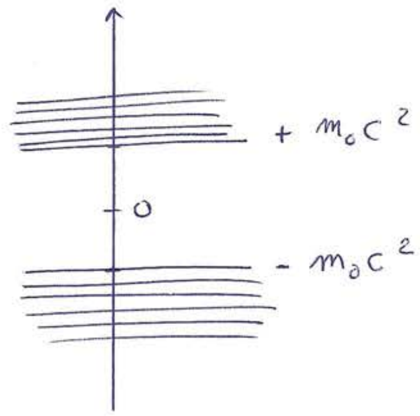
RELATIVISTIC ENERGY - MOMENTUM RELATION

$$\downarrow$$

$$E^2=c^2\bar{p}^2+m_0^2c^4$$

$$E_{\bar{p}}=\pm\sqrt{c^2\bar{p}^2+m_0^2c^4}$$

BOTH POSITIVE & NEGATIVE ENERGY SOLUTIONS



NEGATIVE ENERGY SOLUTIONS WILL BE LINKED WITH ANTI-PARTICLES.

↳ CONSERVED FOUR-CURRENT DENSITY J^μ

$$\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2}\right)\Psi = 0 \qquad (1)$$

$$\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2}\right)\Psi^* = 0 \qquad (2)$$

$$\Psi^* . (1) - \Psi . (2) = 0$$

$$\Updownarrow$$

$$\Psi^* \partial_\mu \partial^\mu \Psi - \Psi \partial_\mu \partial^\mu \Psi^* = 0$$

$$\partial_\mu \left[\Psi^* \partial^\mu \Psi - \Psi \partial^\mu \Psi^*\right] = 0$$

↳

$$\boxed{J^\mu \equiv \frac{i\hbar}{2m_0}\left[\Psi^*(\partial^\mu \Psi) - \Psi(\partial^\mu \Psi^*)\right]}$$

$\partial_\mu J^\mu = 0$ ↳ TO GET SCHRÖDINGER RESULT IN NON-RELATIVISTIC LIMIT

$$J^\mu = (c\rho, \bar{J})$$

$$\partial_\mu J^\mu = 0 \iff \frac{\partial}{\partial x^\mu} J^\mu = 0$$

$$\iff \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \bar{\nabla}.\bar{J} = 0$$

$$\iff \frac{\partial \rho}{\partial t} + \bar{\nabla}.\bar{J} = 0 \qquad \text{(CONTINUITY EQ.)}$$

↳ NON-RELATIVISTIC LIMIT

$$\Psi(\bar{x},t) = \varphi(\bar{x},t)\, e^{-\frac{i}{\hbar}(m_0 c^2)t}$$

SEPARATE OFF
REST MASS TIME DEPENDENCE

$$E = \underset{\substack{\uparrow \\ \text{KINETIC ENERGY}}}{E'} + m_0 c^2$$

NON-RELATIVISTIC $\quad E' \ll m_0 c^2$

e.g. e^- $\quad m_0 c^2 \approx 0.5$ MeV

$$\leadsto \frac{\partial \Psi}{\partial t} = \left[-\frac{i}{\hbar}(m_0 c^2)\varphi + \frac{\partial \varphi}{\partial t}\right] e^{-\frac{i}{\hbar}(m_0 c^2)t}$$

$$\leadsto \frac{\partial^2 \Psi}{\partial t^2} = \left[-\frac{1}{\hbar^2}(m_0 c^2)^2 \varphi - \frac{2i}{\hbar}(m_0 c^2)\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial t^2}\right] \cdot e^{-\frac{i}{\hbar}(m_0 c^2)t}$$

NON-REL. $\left| i\hbar \frac{\partial \varphi}{\partial t} \right| \approx E'\varphi \ll m_0 c^2$

↳ NEGLECT $\frac{\partial^2 \varphi}{\partial t^2}$ TERM (2° ORDER)

$$\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2} \right) \Psi = 0.$$

$$\Updownarrow$$

$$\cancel{-\frac{1}{c^2\hbar^2}(m_0 c^2)^2 \varphi} - \frac{2i}{c^2\hbar}(m_0 c^2)\frac{\partial \varphi}{\partial t} - \nabla^2 \varphi + \cancel{\frac{m_0^2 c^2}{\hbar^2}\varphi} = 0$$

$$\Updownarrow$$

$$\frac{i}{\hbar}\frac{\partial \varphi}{\partial t} = -\frac{\hbar}{2m_0}\nabla^2 \varphi$$

$$\Updownarrow$$

$$\underline{\underline{-\frac{\hbar^2}{2m_0}\nabla^2 \varphi = i\hbar \frac{\partial \varphi}{\partial t}}}$$

NON-RELATIVISTIC SCHRÖDINGER EQ.

2) LAGRANGIAN DENSITY FOR KLEIN-GORDON FIELD

- CLASSICAL LAGRANGIAN FIELD THEORY

↳ SYSTEM SPECIFIED BY FIELDS $\phi_r(x)$

x DENOTES $(ct, \mathbf{x})$

r LABELS COMPONENTS OF FIELD

↳ SCALAR FIELD $(r=1)$ $\phi(x)$

⇝ e.g. TEMPERATURE AT GIVEN POINT IN SPACE & TIME

↳ VECTOR POTENTIAL $(r=1,2,3,4)$ $A^\mu(x)$

$\mu = 0,1,2,3$

↳ DYNAMICS DESCRIBED BY LAGRANGIAN DENSITY

$$\mathcal{L}(\phi_r, \partial^\mu \phi_r) \qquad L = \int d^3\bar{x}\ \mathcal{L}$$

THROUGH VARIATIONAL PRINCIPLE

$$\text{ACTION} \quad S = \int_\Omega d^4x\ \mathcal{L}(\phi_r, \partial^\mu \phi_r)$$

S IS MINIMIZED $\Rightarrow$ FIELD EQUATIONS FOR ϕ_r

FOR VARIATIONS WHICH VANISH AT BOUNDARY $\Gamma(\Omega)$

i.e. $\delta\phi_r = 0$ ON $\Gamma(\Omega)$

FOR $\phi_r \to \phi_r + \delta\phi_r$

$$\delta S' = 0$$

$$\Updownarrow$$

$$\delta S = \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r} \delta\phi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta(\partial_\mu \phi_r) \right\}$$

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) \right\} (\delta\phi_r)$$

$$+ \underbrace{\int_\Omega d^4x \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \cdot \delta\phi_r \right)}_{\text{BOUNDARY TERM}}$$

BUT $\delta\phi_r = 0$ ON $\Gamma(\Omega)$

$$\Downarrow$$

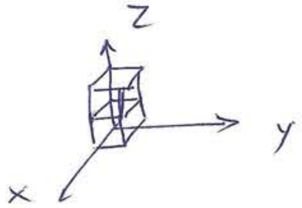
$\delta S = 0$ FOR ARBITRARY $\delta\phi_r$

$$\Updownarrow$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0}$$

EULER - LAGRANGE EQUATIONS.

↳ CONJUGATE MOMENTA / HAMILTONIAN



* $\phi_r(\bar{x}, t)$ FIELD

  DIVIDE SPACE IN CELLS
  POSITION OF CELL : $i = 1, 2, 3 \ldots$
  (DISCRETIZATION)

$$\underbrace{q_{ri}(t)}_{\substack{\uparrow \\ \text{GENERALIZED COORDINATES}}} \equiv \phi_r(i, t) \qquad i = 1, 2, \ldots$$

* $$L = \int d^3\bar{x} \, \mathcal{L}$$

$$= \sum_i (\delta \bar{x}_i) \, . \, \mathcal{L}\left(\phi_r(i,t), \dot{\phi}_r(i,t)\right)$$

* CONJUGATE MOMENTA

$$p_{ri}(t) = \frac{\partial L}{\partial \dot{q}_{ri}} \equiv \frac{\partial L}{\partial \dot{\phi}_r(i,t)} \equiv \delta \bar{x}_i \, . \, \underbrace{\Pi_r(i,t)}_{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(i,t)}}$$

$$\Pi_r(i, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(i,t)}$$

* HAMILTONIAN

$$H = \sum_{r,i} \dot{q}_{ri} \, p_{ri} - L$$

$$= \sum_{r,i} (\delta \bar{x}_i) \, . \left\{ \dot{\phi}_r(i,t) \, \Pi_r(i,t) - \mathcal{L} \right\}$$

$$=: \int d^3\bar{x} \, \mathcal{H}\left(\phi_r(i,t), \Pi_r(i,t)\right)$$

* FIELD THEORY : LIMIT $\delta \bar{x}_i \to 0$.

CONTINUOUS VARIABLE $\phi_r(\bar{x},t)$
(FIELD)

CONJUGATE MOMENTUM FIELD $\boxed{\pi_r(\bar{x},t) \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_r}}$

$$L(t) = \int d^3\bar{x} \quad \mathcal{L}(\phi_r, \partial^\mu \phi_r)$$

$$H(t) = \int d^3\bar{x} \quad \mathcal{H}(\underbrace{\bar{x},t}_{x})$$

$$\boxed{\mathcal{H}(x) = \pi_r(x)\,\dot{\phi}_r(x) - \mathcal{L}}$$

↳ HAMILTONIAN DENSITY

↳ SYMMETRIES & CONSERVATION LAWS

TRANSFORMATION

$$\phi_r(x) \to \phi'_r(x) = \phi_r(x) + \delta\phi_r(x)$$

INDUCES CHANGE $\delta\mathcal{L}$ TO $\mathcal{L}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta(\partial_\mu\phi_r)$$

$$= +\left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\cdot\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu(\delta\phi_r)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\cdot\delta\phi_r\right)$$

IF ABOVE TF. IS SYMM. TRANSFORMATION

⇓

$\delta\mathcal{L} = 0$ (LAGRANGE DENSITY STAYS UNCHANGED)

$$\underline{\underline{\partial_\mu J^\mu = 0}}$$

(NOETHER THEOREM)

CONSERVED CURRENT

$$\boxed{J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r}$$

↓

CONSERVED CHARGE

$$\boxed{\begin{aligned} Q &\equiv \int d^3\bar{x}\; J^0(x) \\ &= c\int d^3\bar{x}\; \Pi_r(x)\,\delta\phi_r(x) \end{aligned}}$$

EXAMPLE : SYMMETRY UNDER SPACE-TIME TRANSLATION

* $x^\mu \to x'^\mu \equiv x^\mu + \underbrace{\delta x^\mu}_{\text{CONSTANT TRANSLATION}}$

* $\underline{\underline{\phi'_r(x') = \phi_r(x)}}$ TRANSFORMED FIELD AT SHIFTED POSITION = ORIGINAL FIELD AT ORIGINAL POSITION

$$\delta\phi_r(x') = \phi'_r(x') - \phi_r(x')$$

$$= -\left[\phi_r(x') - \phi_r(x)\right]$$

$$= -\left(\partial_\mu \phi_r\right)\delta x^\mu$$

* $\mathcal{L}\left(\phi'_r(x'), \partial_\mu \phi'_r(x')\right) = \mathcal{L}\left(\phi_r(x), \partial_\mu \phi_r(x)\right)$

$\Rightarrow \quad 0 = \delta\mathcal{L} + \dfrac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu$ (INVARIANCE UNDER TF.)

DUE TO $\delta\phi_r$

FOR $\mathcal{L}$ DEPENDING ON x^μ

USE $\delta\mathcal{L} = \partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r\right)$

$$0 = \partial_\mu\left(-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial^\nu\phi_r + \mathcal{L}\, g^{\mu\nu}\right)\delta x_\nu$$

$$\underline{\underline{\partial_\mu T^{\mu\nu} = 0}}$$

$$\boxed{T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial^\nu\phi_r - \mathcal{L}\, g^{\mu\nu}}$$

ENERGY-MOMENTUM TENSOR

CONSERVED QUANTITY

$$\boxed{\begin{aligned} c\,P^{\nu} &\equiv \int d^3\overline{x}\;\; T^{0\nu} \\ &= \int d^3\overline{x}\,\left\{ c\,\Pi_r(x)\,\partial^{\nu}\phi_r - \mathcal{L}\,g^{0\nu} \right\} \end{aligned}}$$

P^{ν} is 4-MOMENTUM $\Rightarrow$ 4-MOMENTUM IS CONSERVED

$$\rightsquigarrow P^0 = \frac{1}{c}\int d^3\overline{x}\,\left\{ \Pi_r(x)\,\dot{\phi}_r(x) - \mathcal{L} \right\}$$

$$= \frac{1}{c}\int d^3\overline{x}\;\; \mathcal{H}(x)$$

$$= \frac{1}{c}\,H \qquad (\text{ENERGY})$$

$$\rightsquigarrow \overline{P} = -\int d^3\overline{x}\;\; \Pi_r(x)\left(\overline{\nabla}\,\phi_r\right)$$

$(i = 1, 2, 3)$

(3 - MOMENTUM)

- $\mathcal{L}$ FOR KLEIN-GORDON FIELD $\phi(x)$

↳ $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2 \phi^2$$

↓

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\mu^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (\partial^\mu \phi)$$

⇝ EULER - LAGRANGE EQ. $(\partial_\mu \partial^\mu + \mu^2)\phi = 0$

WITH $\mu^2 = \frac{m_0^2 c^2}{\hbar^2}$

↳ CONJUGATE MOMENTUM

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{c^2}\dot{\phi} \qquad \left(\dot{\phi} \equiv \frac{\partial \phi}{\partial t}\right)$$

3) QUANTIZATION OF KLEIN - GORDON FIELD

- QUANTIZED LAGRANGIAN FIELD THEORY

CANONICAL COMMUTATION RELATIONS

↳ POSTULATE EQUAL TIME COMMUTATION RELATIONS (ETCR) FOR FIELDS & MOMENTA ⇝ WHICH BECOME OPERATORS IN QUANTUM THEORY

$$[\phi_r(\bar{x},t),\ \phi_s(\bar{x}',t)] = 0$$

$$[\Pi_r(\bar{x},t),\ \Pi_s(\bar{x}',t)] = 0$$

$$[\phi_r(\bar{x},t),\ \Pi_s(\bar{x}',t)] = i\hbar\,\delta_{rs}\,\delta^3(\bar{x}-\bar{x}')$$

SAME TIME t !

↳ FOR REAL KLEIN - GORDON FIELD (DESCRIBING NEUTRAL SPIN-0 PARTICLES)

$$[\phi(\bar{x},t),\ \phi(\bar{x}',t)] = 0$$

$$[\dot{\phi}(\bar{x},t),\ \dot{\phi}(\bar{x}',t)] = 0$$

$$[\phi(\bar{x},t),\ \dot{\phi}(\bar{x}',t)] = i\hbar c^2\,\delta^3(\bar{x}-\bar{x}')$$

↳ NORMAL MODE EXPANSION FOR ϕ : (FOR $\phi^+ = \phi$) (ANALOGOUS TO E.M. FIELD CONSIDERED BEFORE, no σ)

$$\boxed{\phi(\bar{x},t) = \sum_{\bar{k}} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \left\{ a(\bar{k})\, e^{-i\omega_k t + i\bar{k}\cdot\bar{x}} + a^+(\bar{k})\, e^{i\omega_k t - i\bar{k}\cdot\bar{x}} \right\}}$$

$a(\bar{k})$: ANNIHILATION OPERATOR
OF SPIN-0 PARTICLE WITH MOMENTUM $\hbar \bar{k}$
ENERGY $\hbar \omega_k$

$a^+(\bar{k})$: CREATION OPERATOR
OF SPIN-0 PARTICLE WITH MOMENTUM $\hbar \bar{k}$
ENERGY $\hbar \omega_k$

$$\hookrightarrow \; i\,\dot{\phi}(x,t) = \sum_{\bar{k}} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \omega_k \left\{ a(\bar{k})\, e^{-i\omega_k t + i\bar{k}.\bar{x}} - a^+(\bar{k})\, e^{i\omega_k t - i\bar{k}.\bar{x}} \right\}$$

$$\int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}}\, \phi(\bar{x},t) = \sum_{\bar{k}'} \left(\frac{\hbar c^2}{2\omega_{k'} L^3}\right)^{1/2} \left\{ a(\bar{k}')\, e^{-i\omega_{k'} t} L^3 \delta_{\bar{k}'\bar{k}} + a^+(\bar{k}')\, e^{i\omega_{k'} t} L^3 \delta_{\bar{k}'-\bar{k}} \right\}$$

$$\int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}}\, i\dot{\phi}(\bar{x},t) = \sum_{\bar{k}'} \left(\frac{\hbar c^2}{2\omega_{k'} L^3}\right)^{1/2} \omega_{k'} \left\{ a(\bar{k}')\, e^{-i\omega_{k'} t} L^3 \delta_{\bar{k}'\bar{k}} - a^+(\bar{k}')\, e^{i\omega_{k'} t} L^3 \delta_{\bar{k}'-\bar{k}} \right\}$$

$$\Downarrow$$

$$\int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}} \left\{ \omega_k \phi + i\dot{\phi} \right\} = 2\left(\frac{\hbar c^2}{2\omega_k}\right)^{1/2} (L^3)^{1/2}\, \omega_k\, a(\bar{k})\, e^{-i\omega_k t}$$

$$\int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}} \left\{ \omega_k \phi - i\dot{\phi} \right\} = 2\left(\frac{\hbar c^2}{2\omega_k}\right)^{1/2} (L^3)^{1/2}\, \omega_k\, a^+(-\bar{k})\, e^{i\omega_k t}$$

$$\boxed{a(\bar{k}) = \frac{1}{(2\hbar c^2 \omega_k L^3)^{1/2}} \int d^3\bar{x}\; e^{ik.x} \left\{ \omega_k \phi(\bar{x},t) + i\dot{\phi}(\bar{x},t) \right\}}$$

NOTE $e^{ikx} = e^{i\omega_k t - i\bar{k}.\bar{x}}$

$$\boxed{a^+(\bar{k}) = \frac{1}{(2\hbar c^2 \omega_k L^3)^{1/2}} \int d^3\bar{x}\; e^{-ik.x} \left\{ \omega_k \phi(\bar{x},t) - i\dot{\phi}(\bar{x},t) \right\}}$$

(CHANGE $-\bar{k} \to \bar{k}$ IN PREVIOUS)

↳ COMMUTATION RELATIONS FOR a, a^+

USE ETCR FOR FIELDS TO OBTAIN

$$[a(\bar{k}), a^+(\bar{k}')] = \delta_{\bar{k}\bar{k}'}$$

$$[a(\bar{k}), a(\bar{k}')] = 0$$

$$[a^+(\bar{k}), a^+(\bar{k}')] = 0$$

$N(\bar{k}) = a^+(\bar{k})\, a(\bar{k})$ NUMBER OPERATOR

HAMILTONIAN FOR KLEIN - GORDON FIELD

$$\mathcal{H}(x) = \pi \dot{\phi} - \mathcal{L}$$

$$= \frac{1}{c^2} \dot{\phi}^2 - \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} \mu^2 \phi^2$$

$$= \frac{1}{2} \left[\frac{1}{c^2} \dot{\phi}^2 + (\bar{\nabla} \phi)^2 + \mu^2 \phi^2 \right]$$

$$H = \int d^3 \bar{x} \; \mathcal{H}$$

SHOW THAT (EXERCISE)

$$H = \sum_{\bar{k}} \hbar \omega_k \left(a^+(\bar{k})\, a(\bar{k}) + \frac{1}{2} \right)$$

MOMENTUM

$$\bar{P} = - \int d^3 \bar{x} \; \frac{1}{c^2} \dot{\phi} \, \bar{\nabla} \phi$$

SHOW THAT

$$\bar{P} = \sum_{\bar{k}} \hbar \bar{k} \cdot \left(a^+(\bar{k})\, a(\bar{k}) \right)$$