Practice Exam
Theoretical Physics 3: QM WS2022/2023
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Exercise 1. General questions (20 points + 10 bonus)

1.1. *(5 p.)* Consider the ground state wave function of the harmonic oscillator in spatial representation

$$\langle x|\psi_0\rangle = A_0\, e^{-\frac{m\omega}{2\hbar}x^2}.$$

Recall

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px} \quad \text{and} \quad \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

Compute $\langle p|\psi_0\rangle$.

1.2. *(5 p.)* Consider an operator $\hat{T}(a) \equiv e^{\frac{ia}{\hbar}\hat{p}}$, where $\hat{p}$ is the momentum operator and a is a real parameter.

a) Is it an observable? Why?

b) Show that $\hat{T}(a)\,\psi(x) = \psi(x+a)$.

1.3. *(5 p.)* Assume $\hat{H}$ is the *time-independent* Hamiltonian.

a) Show that the operator $\hat{U}(t-t_0) \equiv e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$ is unitary.

b) Show that the solution to the time-dependent Schrödinger equation is

$$\Psi(x,t) = \hat{U}(t-t_0)\Psi(x,t_0),$$

with $\Psi(x,t_0)$ being a given wave function of the system at time t_0.

1.4. *(5 p.)* Consider two observables $\hat{A}$ and $\hat{B}$.
$\hat{A}$ has two normalized eigenstates $|a_1\rangle$ and $|a_2\rangle$, with eigenvalues a_1 and a_2, respectively.
$\hat{B}$ has two normalized eigenstates $|b_1\rangle$ and $|b_2\rangle$, with eigenvalues b_1 and b_2, respectively.
Assume the eigenstates are related by

$$|a_1\rangle = \frac{3}{5}|b_1\rangle + \frac{4}{5}|b_2\rangle\,, \qquad |a_2\rangle = \frac{4}{5}|b_1\rangle - \frac{3}{5}|b_2\rangle\,.$$

a) The observable $\hat{A}$ is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

b) If afterwards $\hat{B}$ is measured, what are the possible outcomes, and what are their probabilities?

c) Right after $\hat{B}$ is measured, $\hat{A}$ is measured again. What is the probability of getting a_1?

1.5. *(Bonus 10 p.) Eigenfunctions and degeneracy.*

a) *(2 p.)* What is the degree of degeneracy for the energy of a one-dimensional free particle?

b) *(3 p.)* Is the ground state of an infinite square well an eigenfunction of momentum? If so, what is its momentum? If not, why not?

c) *(5 p.)* Using the Schrödinger equation, prove that in one dimension there are no degenerate bound states.

Exercise 2. Half-harmonic oscillator (30 points + 5 bonus)

Consider a particle of mass m, which is moving in one dimension in a "half"-harmonic potential $V(x)$

$$V(x) = \begin{cases} \infty, & x < 0; \\ \frac{1}{2} m\omega^2 x^2, & x \geq 0. \end{cases}$$

a) *(5 p.)* Write down the *stationary* Schrödinger equation for $x \geq 0$ using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}} x \quad \text{and} \quad \varepsilon = \frac{E}{\hbar\omega}.$$

b) *(5 p.)* Show that the asymptotic behavior of the solution for large y is given by $e^{-y^2/2}$.

c) *(8 p.)* By separating the asymptotic behavior for $y \to \infty$, we define

$$\psi(y) = h(y) e^{-y^2/2}.$$

Derive the equation for $h(y)$ for $y \geq 0$.

d) *(12 p.)* We know that for the *regular* quantum harmonic oscillator the eigenfunctions of the Hamiltonian are expressed in terms of the Hermite polynomials:

$$\psi_n(y) \propto H_n(y) e^{-y^2/2}, \quad n = 0, 1, 2, \dots ,$$

where the Hermite polynomials $H_n(y)$ satisfy the differential equation

$$H_n''(y) - 2y H_n'(y) + 2n H_n(y) = 0, \quad n = 0, 1, 2, \dots ,$$

and can equivalently be defined as

$$H_n(y) = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2}.$$

Deduce the spectrum in the case of the given *"half"*-harmonic potential.

e) *(Bonus 5 p.)* The Hermite polynomials are normalised as

$$\int_{-\infty}^{\infty} dy\, H_n(y) H_m(y) e^{-y^2} = 2^n n! \sqrt{\pi} \delta_{nm}.$$

What are the normalised *ground* state and *first excited* state wave functions of the given *"half"*-harmonic potential?

Exercise 3. Stark effect (25 points)

Consider an electron in the $n = 2$ state of the hydrogen atom. The electric dipole moment $\vec{d} = -e\vec{r}$ of the electron interacts with an external electric field $\vec{E}$ through

$$\hat{H}_E' = -\vec{d} \cdot \vec{E},$$

which can be treated as a perturbation to the Coulomb potential.
Assume a constant electric field along the x-axis:

$$\vec{E} = E_0 \, \vec{e}_x.$$

We denote the unperturbed eigenstates $|n\, l\, m_l\rangle$ (neglecting spin) as

$$\begin{aligned} |1\rangle &\equiv |2\,0\,0\rangle, \\ |2\rangle &\equiv |2\,1\,0\rangle, \\ |3\rangle &\equiv |2\,1\,+1\rangle, \\ |4\rangle &\equiv |2\,1\,-1\rangle. \end{aligned}$$

a) *(15 p.)* Recall the hydrogen atom wave functions are given by

$$\psi_{nlm_l}(r,\theta,\phi) = R_{n,l}(r)\, Y_{l,m_l}(\theta,\phi).$$

You are given the spherical harmonics

$$\begin{aligned} Y_{0,0}(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}, \\ Y_{1,0}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta, \\ Y_{1,\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}, \end{aligned}$$

and the radial integral

$$\int_0^\infty dr\, r^3 R_{2,0}(r) R_{2,1}(r) = 3\sqrt{3}\, a,$$

with a being the Bohr radius.

Determine the 4×4 matrix form of $\hat{H}'_E$ in the unperturbed basis in terms of $\Omega_e \equiv eE_0\frac{a}{\hbar}$.

Hint: use symmetry relations to argue that several of the angular integrals are zero.

b) *(10 p.)* Diagonalize the above matrix to calculate the first order corrections to all four $n=2$ levels due to $\hat{H}'_E$ (you only need to find the eigenvalues, not the eigenstates).

Make a qualitative sketch of the total energy of the $n=2$ levels as a function of the externally applied electric field E_0. Comment on their degeneracies.

Exercise 4. A particle with spin $3/2$ (25 points)

Consider a particle with spin $3/2$.

a) *(3 p.)* Find the basis $|s,s_z\rangle$ of eigenstates of S_z for such a particle.

b) *(8 p.)* Using the previous basis, find the matrices representing S_x and S_y.
Hint: use $S_\pm = S_x \pm iS_y$ and
$S_\pm|s,s_z\rangle = \hbar\sqrt{s(s+1) - s_z(s_z\pm 1)}|s,s_z\pm 1\rangle$.

c) *(8 p.)* Solve the characteristic equation to determine the eigenvalues of S_x.

d) *(6 p.)* Write down, at least, one eigenvector of S_x.