# Exercise sheet 10
## Theoretical Physics 3: QM WS2022/2023
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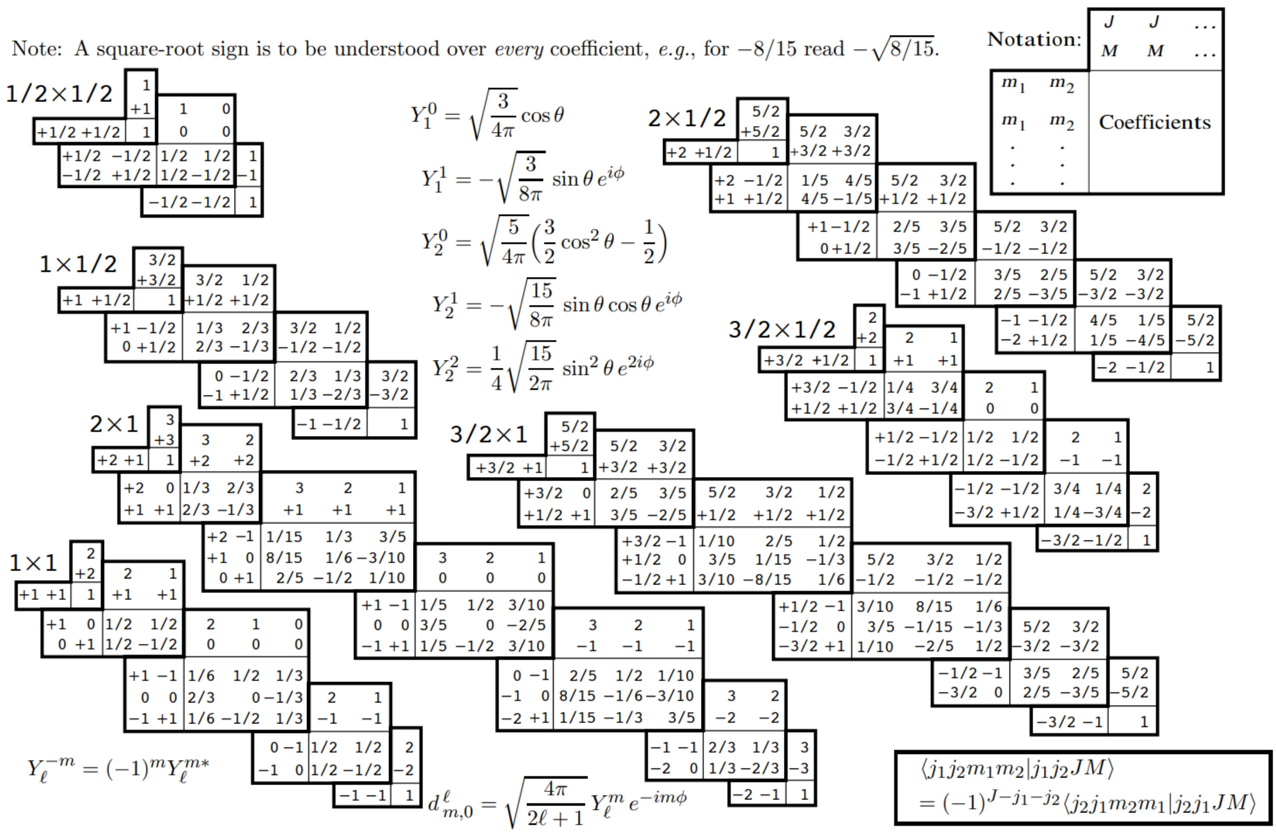
## Exercise 1. Clebsch-Gordan coefficients (40 points)

In this exercise we will practice how to couple two angular momenta $j_1$ and $j_2$, using the Clebsch-Gordan Table.
Recall that the coupled states which are characterized by the total angular momentum $J$ and its projection $M$ can be expanded via the completeness relation in the uncoupled basis:

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2|JM\rangle$$

The expansion coefficients, $\langle j_1 m_1 j_2 m_2|JM\rangle$, are the Clebsch-Gordan coefficients which can be found in the following table:




a) *(25 p.)* Write down all the possible states $|JM\rangle$ in the basis $|j_1 m_1\rangle\,|j_2 m_2\rangle$ for the compositions $\frac{1}{2}\otimes 1$ and $1\otimes 1$ (the symbol $\otimes$ stands for the coupling of two angular momenta).

b) *(15 p.)* Check explicitly that the decompositions of the state $|\frac{5}{2},+\frac{1}{2}\rangle$ in the basis $|\frac{1}{2}m_1\rangle\,|1m_2\rangle\,|1m_3\rangle$ obtained from $(\frac{1}{2}\otimes 1)\otimes 1$ and $\frac{1}{2}\otimes(1\otimes 1)$ are the same.

## Exercise 2. Spin 1/2 states (30 points)

Consider a general spin-1/2 state:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

Which is normalized, $|a|^2 + |b|^2 = 1$.

a) *(10 p.)* Show that there always exists a direction in space $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ such that $\chi$ is the eigenstate of the spin component along this direction $S_{\vec{n}} = \vec{n}\cdot\vec{S}$ with eigenvalue $\hbar/2$.

b) *(15 p.)* Write $\theta$ and $\phi$ in terms of $a$ and $b$.

c) *(5 p.)* Would an analogous result hold for higher spin states?

*Hint*: Count the number of degrees of freedom.

## Exercise 3. Spin 1 matrices (30 points)

a) *(15 p.)* Derive the spin matrices $S_x, S_y, S_z$ in the basis $|s, s_z\rangle$ for $s = 1$.

b) *(15 p.)* Find the eigenvalues and the normalized eigenvectors of $S_x$ and $S_y$ in that basis.

*Hint*: The general relation $S_\pm|s, s_z\rangle = \hbar\sqrt{s(s+1) - s_z(s_z \pm 1)}|s, s_z \pm 1\rangle$ can we useful.