Exercise sheet 9
Theoretical Physics 3: QM WS2022/2023
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Exercise 1. Infinite spherical well (30 points)

Consider a particle in an infinite well 3D potential of radius a:

$$V(r) = \begin{cases} 0, & r < a \\ +\infty, & r \geq a \end{cases}$$

a) *(15 p.)* Show that the solution of the Schrödinger equation is:

$$\Psi_{nlm}(r, \theta, \phi) \propto j_l\left(\beta_{nl}\frac{r}{a}\right) Y_{lm}(\theta, \phi)$$

And $j_l(x)$ is the spherical Bessel functions of order l which is defined as:

$$j_l(x) \equiv (-x)^l \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sin x}{x}$$

$j_l(x)$ is the non-singular at zero solution of the differential equation:

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left[x^2 - l(l+1)\right] y = 0$$

β_{nl} is the n-th zero of the a spherical Bessel functions of order l: $j_l(\beta_{nl}) = 0$.

b) *(15 p.)* The spherical Bessel functions is a particular case of the Bessel functions $J_\alpha(x)$ defined as:

$$J_\alpha = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(l+\alpha+1)} \left(\frac{x}{2}\right)^{2l+\alpha}$$

For α being half-integer, so $J_{l+1/2} = \sqrt{\frac{2x}{\pi}}\, j_l(x)$.
Using the definition of the Bessel functions, compute $J_{1/2}$ and $J_{3/2}$ and check that, indeed, the relation between $J_{l+1/2}$ and j_l is correct.
Hint: prove $l!(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2l+1))2^n = (2l+1)!$.

Math hints:

$$\sin(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

$$\Gamma(m+1/2) = \frac{1 \cdot 3 \cdot 5 \cdot (2m-1)}{2^m}\sqrt{\pi}$$

Exercise 2. Hydrogen atom (20 + 10 points)

The normalized hydrogen wave functions are:

$$\psi_{nlm}(r,\theta,\phi) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{[a(n+l)!]^3}}\, e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}(\tfrac{2r}{na})\, Y_l^m(\theta,\phi)$$

Where $L^p_{q-p}(x)$ are the associated Laguerre polynomials and $Y_l^m(\theta,\phi)$ are the spherical harmonics.

a) *(5 p.)* Consider the electron is in the state $\psi_{nlm}(r,\theta,\phi)$. What is the probability $P_{nl}(r)$ to find it somewhere?

b) *(15 p.)* Check explicitly that $P_{nl}(r)$ is correctly normalized to unity for $n=3$.
Hint: use $\int_0^\infty \mathrm{d}x\, e^{-x}x^n = n!$.

c) *(10 p.)* *(Bonus)* Show that $\int_0^\infty \mathrm{d}x\, e^{-x}x^n = n!$.

Exercise 3. 2D quantum harmonic oscillator (50 + 20 points)

a) *(10 p.)* Assuming solutions for the one-dimensional case are already known, solve the two-dimensional isotropic quantum harmonic oscillator problem in the Cartesian coordinates:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y) + \frac{m\omega^2}{2}\left(x^2+y^2\right)\psi(x,y) = E\psi(x,y)$$

Hint: use the method of separation of variables: $\psi(x,y) = X(x)Y(y)$. Then write down separate equations on $X(x)$ and $Y(y)$.

b) *(5 p.)* Write down the energy spectrum. What is the degree of degeneracy of the energy levels?

c) *(10 p.)* Show that the Laplace operator in two dimensions in the polar coordinates takes the form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$$

d) *(10 p.)* Write down the angular momentum operator $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ in the polar coordinates and show that $[\hat{H},\hat{L}_z] = 0$.

e) *(10 p.)* Consider the two-dimensional isotropic harmonic oscillator in polar coordinates. Separate variables $\psi(r,\phi) = v(r)u(\phi)$ and get equations on $v(r)$ and $u(\phi)$.

The equation on $v(r)$ can be eventually transformed into the one for the generalised Laguerre polynomials $L^{|M|+1}_{n_r}\left(\frac{m\omega}{\hbar}r^2\right)$. Then one obtains the final solution:

$$\psi_{n_r M}(r,\phi) = C_{n_r M}\, r^{|M|} e^{\frac{-m\omega}{2\hbar}r^2} L^{|M|+1}_{n_r}\left(\frac{m\omega}{\hbar}r^2\right) e^{iM\phi}$$

With the spectrum:

$$E = \hbar\omega(|M|+1+2n_r), \quad n_r = 0,1,2,\ldots, \quad M = 0,\pm1,\pm2,\ldots$$

Where M is the quantum number corresponding to $\hat{L}_z$.

f) *(5 p.)* Find eigenvalues and eigenfunctions of $\hat{L}_z$ in polar coordinates. Show that, indeed, the complete and orthonormal set of eigenfunctions is common to both $\hat{H}$ and $\hat{L}_z$.

g) *(20 p.)* *(Bonus)* Find the ground state solution of the Schrödinger equation ($n_r = 0, M = 0$) in polar coordinates.
Hint: put $E = \hbar\omega$, $u''(\phi) = 0$ and substitute $v(r) = e^{-\frac{m\omega}{2\hbar}r^2}F(r)$.