Exercise sheet 8
Theoretical Physics 3: QM WS2022/2023
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Exercise 1. Laplace operator in spherical coordinates (20 points)

a) *(15 p.)* Show that the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three dimensions in spherical coordinates takes the form:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

b) *(5 p.)* Show that the radial term can also be written as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2} r$$

Exercise 2. Angular momentum operator (45 points)

a) *(15 p.)* Show that $L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_l^{m\pm 1}$.
Hint: consider the norm of $L_\pm Y_l^m$.

b) *(15 p.)* Show that for eigenfunctions of $\hat{L}_z$, we have

$$\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = 0; \quad \langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle; \quad \langle \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \rangle = 0$$

Hint: consider $\langle \hat{L}_\pm \rangle$ and $\langle \hat{L}_\pm^2 \rangle$.

c) *(15 p.)* In the state ψ_{lm} with definite angular momentum l and its z-component m, find the mean values $\langle \hat{L}_x^2 \rangle$, $\langle \hat{L}_y^2 \rangle$ as well as the mean values $\langle \hat{L}_{\tilde{z}} \rangle$ and $\langle \hat{L}_{\tilde{z}}^2 \rangle$ of the angular momentum projection along the $\tilde{z}$-axis making an angle α with the z-axis.
Hint: use $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}^2$.

Exercise 3. Expectation values (35 points)

a) *(20 p.)* Prove that for a particle in a potential $V(\vec{r})$ the rate of change of the expectation value of the orbital angular momentum $\vec{L}$ is equal to the expectation value of the torque (rotational analog to Ehrenfest's theorem):

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \vec{L} \rangle = \langle \vec{N} \rangle; \quad \vec{N} = \vec{r} \times (-\vec{\nabla} V)$$

Show that $\langle \vec{L} \rangle$ is constant for any spherically symmetric potential. (This is one form of the quantum statement of conservation of angular momentum).

b) *(15 p.)* Show that the mean values of the vectors $\vec{L}$, $\vec{r}$, $\vec{p}$ for the particle state with wave function $\psi = \exp(i\vec{p}_0 \cdot \vec{r}/\hbar)\phi(\vec{r})$ are connected by the classical relation $\vec{L} = \vec{r} \times \vec{p}$. Here, p_0 is a real vector and $\phi(\vec{r})$ is a real function.

(Bonus) Exercise 4. Degenerate quantum numbers (15 points)

Prove or disprove the following statements:

a) *(10 p.)* If $[\hat{H}, \hat{\vec{L}}] = \vec{0}$, the energy levels do not depend on m (i.e. on the eigenvalues of the projection of one of the components of the angular momentum $\hat{\vec{L}}$).

b) *(5 p.)* If $[\hat{H}, \hat{L}^2] = 0$, the energy levels do not depend on l.