Joachain

Appendix B

Legendre Polynomials, Associated Legendre Functions and Spherical Harmonics

1. Legendre polynomials

Let us consider the real variable x such that $-1 \leqslant x \leqslant +1$. We may also set $x = \cos\theta$, where θ is a real number. The polynomials of degree l ($l = 0, 1, 2, \ldots$)

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2 - 1)^l \tag{B.1}$$

are known as the *Legendre polynomials.* They satisfy the differential equation

$$\left[(1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1)\right]P_l(x) = 0. \tag{B.2}$$

Furthermore, $P_l(x)$ has the parity $(-)^l$ and has l zeros in the interval $(-1, +1)$. A generating function for the Legendre polynomials is

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = \sum_{l=0}^{\infty} P_l(x)t^l, \qquad |t| < 1. \tag{B.3}$$

One also has the recurrence relations

$$(2l+1)xP_l - (l+1)P_{l+1} - lP_{l-1} = 0, \tag{B.4a}$$

$$(x^2 - 1)\frac{\mathrm{d}P_l}{\mathrm{d}x} = l(xP_l - P_{l-1}) = \frac{l(l+1)}{2l+1}(P_{l+1} - P_{l-1}) \tag{B.4b}$$

(also valid for $l = 0$ if one defines $P_{-1} = 0$). The orthogonality relations read

$$\int_{-1}^{+1} P_l(x)P_{l'}(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{ll'}. \tag{B.5}$$

One also has the closure relation

$$\tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)P_l(\cos\theta') = \delta(\cos\theta - \cos\theta'). \tag{B.6}$$

Important particular values of the Legendre polynomials are

$$P_l(1) = 1, \qquad P_l(-1) = (-1)^l. \tag{B.7}$$

For the lowest values of l one has explicitly

$$\begin{aligned} P_0(x) &= 1,\\ P_1(x) &= x,\\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1)\\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x)\\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned} \tag{B.8}$$

2. Associated Legendre functions

These functions are defined by the relations

$$P_l^m(x) = (1 - x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x), \qquad m = 0, 1, 2, \ldots l, \tag{B.9}$$

and we see that they are the product of the quantity $(1 - x^2)^{m/2}$ and of a polynomial of degree $(l - m)$ and parity $(-)^{l-m}$, having $(l - m)$ zeros in the interval $(-1, +1)$. The functions P_l^m satisfy the differential equation

$$\left[(1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - \frac{m^2}{1 - x^2}\right]P_l^m(x) = 0 \tag{B.10}$$

and they are given from a generating function as

$$(2m-1)!!(1 - x^2)^{m/2}\frac{t^m}{(1 - 2xt + t^2)^{m+1/2}} = \sum_{l=m}^{\infty} P_l^m(x)t^l, \qquad |t| < 1$$

with

$$(2m-1)!! = 1\cdot 3\cdot 5\cdots(2m-1). \tag{B.11}$$

In particular, one has

$$P_l^0(x) = P_l(x), \tag{B.12}$$

$$P_l^l(x) = (2l-1)!!(1 - x^2)^{l/2}. \tag{B.13}$$

The functions P_l^m satisfy the recurrence relations

$$(2l+1)xP_l^m - (l-m+1)P_{l+1}^m - (l+m)P_{l-1}^m = 0, \tag{B.14}$$

$$\begin{aligned}(x^2 - 1)\frac{\mathrm{d}P_l^m}{\mathrm{d}x} &= -(l+1)xP_l^m + (l-m+1)P_{l+1}^m\\ &= lxP_l^m - (l+m)P_{l-1}^m, \qquad 0 \leqslant m \leqslant l-1, \end{aligned} \tag{B.15}$$

$$P_l^{m+2} - 2(m+1)\frac{x}{(1-x^2)^{1/2}}P_l^{m+1} + (l-m)(l+m+1)P_l^m = 0,$$
$$0 \leqslant m \leqslant l-2, \qquad \text{(B.16)}$$

$$P_{l-1}^m - P_{l+1}^m = -(2l+1)(1-x^2)^{1/2}P_l^{m-1}, \qquad 0 \leqslant m \leqslant l-1 \qquad \text{(B.17)}$$

and the orthonormality relations

$$\int_{-1}^{+1} P_l^m(x)P_{l'}^m(x)\,\mathrm{d}x = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}. \qquad \text{(B.18)}$$

Important particular values are

$$P_l^m(1) = P_l^m(-1) = 0, \qquad m \neq 0 \qquad \text{(B.19)}$$

[for $m = 0$, see eq. (B.7)]

$$P_l^m(0) = \begin{cases} (-)^s \dfrac{(2s+2m)!}{2^l s!(s+m)!}, & \text{if } l-m = 2s \\ 0, & \text{if } l-m = 2s+1. \end{cases} \qquad \text{(B.20)}$$

The first few associated Legendre functions are given explicitly by

$$\begin{aligned} P_1^1(x) &= (1-x^2)^{1/2}, \\ P_2^1(x) &= 3(1-x^2)^{1/2}x, \\ P_2^2(x) &= 3(1-x^2), \\ P_3^1(x) &= \tfrac{3}{2}(1-x^2)^{1/2}(5x^2-1), \\ P_3^2(x) &= 15x(1-x^2), \\ P_3^3(x) &= 15(1-x^2)^{3/2}. \end{aligned} \qquad \text{(B.21)}$$

3. *Spherical harmonics*

The spherical harmonics $Y_{lm}(\theta, \phi)$ are eigenfunctions of the operators L^2 and L_z. That is,

$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}, \qquad l = 0, 1, 2, \ldots \qquad \text{(B.22)}$$

$$L_z Y_{lm} = m\hbar Y_{lm}, \qquad m = -l, -l+1, \ldots, l \qquad \text{(B.23)}$$

with

$$L_x = \mathrm{i}\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \qquad \text{(B.24)}$$

$$L_y = -\mathrm{i}\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right), \qquad \text{(B.25)}$$

$$L_z = -\mathrm{i}\hbar\frac{\partial}{\partial\phi} \qquad \text{(B.26)}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \qquad \text{(B.27)}$$

One has [1]

$$Y_{lm}(\theta, \phi) = (-1)^m\left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \qquad m \geqslant 0 \qquad \text{(B.28)}$$

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi). \qquad \text{(B.29)}$$

The functions Y_{lm} have the parity $(-)^l$. Thus, in a reflection about the origin such that $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$, one has

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-)^l Y_{lm}(\theta, \phi). \qquad \text{(B.30)}$$

We also note that for $m = 0$ and $m = l$ the spherical harmonics are given respectively by the simple expressions

$$Y_{l,0}(\theta) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta) \qquad \text{(B.31)}$$

and

$$Y_{l,l}(\theta, \phi) = (-1)^l\left[\frac{2l+1}{4\pi}\frac{(2l)!}{2^{2l}(l!)^2}\right]^{1/2}\sin^l\theta\,\mathrm{e}^{\mathrm{i}l\phi}. \qquad \text{(B.32)}$$

The spherical harmonics satisfy the recurrence relations

$$\begin{aligned} L_\pm Y_{lm} &= \hbar[l(l+1) - m(m\pm 1)]^{1/2} Y_{l,m\pm 1}, \\ L_+ Y_{l,l} &= 0 \\ L_- Y_{l,-l} &= 0 \end{aligned} \qquad \text{(B.33)}$$

with

$$L_\pm = L_x \pm \mathrm{i}L_y = \hbar\,\mathrm{e}^{\pm\mathrm{i}\phi}\left[\pm\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\right]. \qquad \text{(B.34)}$$

The orthonormality relations are

$$\int Y_{l'm'}^*(\theta, \phi)Y_{lm}(\theta, \phi)\,\mathrm{d}\Omega = \int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \mathrm{d}\theta\sin\theta\, Y_{l'm'}^*(\theta, \phi)Y_{lm}(\theta, \phi)$$
$$= \delta_{ll'}\delta_{mm'}, \qquad (\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi) \qquad \text{(B.35)}$$

while the closure relation reads

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_{lm}^*(\theta, \phi)Y_{lm}(\theta', \phi') = \delta(\Omega - \Omega') \qquad \text{(B.36)}$$

with

$$\delta(\Omega - \Omega') = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta}. \qquad \text{(B.37)}$$

The first few spherical harmonics are given by

$$\begin{aligned} Y_{0,0} &= (4\pi)^{-1/2}, \\ Y_{1,0} &= \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \\ Y_{1,1} &= -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}, \end{aligned}$$

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$$
\begin{aligned}
Y_{2,0} &= \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1),\\
Y_{2,1} &= -\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta\, e^{i\phi},\\
Y_{2,2} &= \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{2i\phi},\\
Y_{3,0} &= \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta),\\
Y_{3,1} &= -\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1)\, e^{i\phi},\\
Y_{3,2} &= \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta\, e^{2i\phi},\\
Y_{3,3} &= -\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta\, e^{3i\phi}.
\end{aligned}
\tag{B.38}
$$

4. *Some useful formulae*

If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are two vectors having polar angles (θ_1, ϕ_1) and (θ_2, ϕ_2), and if we denote by θ the angle between these two vectors, the "addition theorem" of spherical harmonics states that

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \tag{B.39a}$$

or

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2) \tag{B.39b}$$

where $\hat{x}$ denotes the polar angles of a vector $\boldsymbol{x}$.

Other useful relations are

$$\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \sum_{l=0}^{\infty} \frac{(r_<)^l}{(r_>)^{l+1}} P_l(\cos\theta) \tag{B.40}$$

or

$$\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} \frac{(r_<)^l}{(r_>)^{l+1}} Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2) \tag{B.41}$$

where $r_<$ is the smaller and $r_>$ the larger of r_1 and r_2. One also has

$$\frac{\exp\{ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|\}}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = ik \sum_{l=0}^{\infty} (2l+1)\, j_l(kr_<)\, h_l^{(1)}(kr_>)\, P_l(\cos\theta) \tag{B.42}$$

or

$$\frac{\exp\{ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|\}}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = 4\pi ik \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y^*_{lm}(\hat{r}_1)\, Y_{lm}(\hat{r}_2) \tag{B.43}$$

where j_l and $h_l^{(1)}$ are respectively the spherical Bessel function and the spherical Hankel function of the first kind (see Appendix C).

The development in spherical harmonics of a plane wave $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ of wave vector $\boldsymbol{k}$ is given by

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} i^l j_l(kr)\, Y^*_{lm}(\hat{k})\, Y_{lm}(\hat{r}). \tag{B.44}$$

Using the addition theorem (B.39), we may also write

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr)\, P_l(\cos\theta) \tag{B.45}$$

where θ is the angle between the directions of the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$. In particular, if we choose the z-axis to coincide with the direction of $\boldsymbol{k}$, we have

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) \equiv e^{ikz} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr)\, P_l(\cos\theta). \tag{B.46}$$

Finally, we quote the relation

$$
\begin{aligned}
&\int Y_{l_1,m_1}(\theta,\phi) Y_{l_2,m_2}(\theta,\phi) Y_{l_3,m_3}(\theta,\phi)\, d\Omega\\
&= \left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}
\end{aligned}
\tag{B.47}
$$

where we have introduced the Wigner 3-j symbols (see Appendix E). From eq. (B.47) one also finds that

$$
\begin{aligned}
Y_{l_1,m_1}(\theta,\phi) Y_{l_2,m_2}(\theta,\phi) &= \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^{L} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}\right]^{1/2}\\
&\quad \times \langle l_1 l_2 00|L0\rangle \langle l_1 l_2 m_1 m_2|LM\rangle Y_{L,M}(\theta,\phi)
\end{aligned}
\tag{B.48}
$$

where we have used vector addition coefficients (see Appendix E). This last equation may also be written in terms of Wigner 3-j symbols as

$$
\begin{aligned}
Y_{l_1,m_1}(\theta,\phi) Y_{l_2,m_2}(\theta,\phi) &= \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^{+L} (-1)^M\\
&\times \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l_1 & l_2 & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & L\\ m_1 & m_2 & M\end{pmatrix} Y_{L,-M}(\theta,\phi).
\end{aligned}
\tag{B.49}
$$

Additional useful formulae involving the Legendre polynomials, associated Legendre functions and spherical harmonics may be found in the references [2–5]. [redacted]

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References and notes

[1] The equations (B.22)–(B.23) and (B.35) determine the functions $Y_{lm}(\theta, \phi)$ up to a phase. The choice of phase made in writing down the relations (B.28)–(B.29) ensures that

i) The functions Y_{lm} obtained in this way verify the recurrence relations (B.33)

ii) $Y_{l,0}(0, 0)$ is real and positive.

Since different authors choose different phase factor conventions for the spherical harmonics, one should be careful to check this point in dealing with the functions Y_{lm} used in the physics literature.

[2] ABRAMOWITZ, M. and I. A. STEGUN (1965), *Handbook of Mathematical Functions* (Dover Publ., New York) Chapter 8.

[3] MAGNUS, W. and F. OBERHETTINGER (1954), *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York) Chapter 4.

[4] ERDELYI, A., W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI (1953), *Higher Transcendental Functions* (Bateman Manuscript Project, McGraw-Hill, New York) Vol. 1, Chapter 3.

[redacted]

Appendix C

Spherical Bessel Functions

Let us consider the differential equation (4.20), namely

$$\left[\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{l(l+1)}{z^2}\right)\right] f_l = 0 \tag{C.1}$$

with $l = 0, 1, 2, \ldots$ Particular solutions of this equation are:

i) The (genuine) *spherical Bessel functions* (or spherical Bessel functions of the first kind)

$$j_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{l+1/2}(z) \tag{C.2}$$

where $J_\nu(z)$ is an ordinary Bessel function of order ν. The functions $j_l(z)$ are *regular* at the origin [see eq. (C.11a)].

ii) *The spherical Neumann functions*

$$n_l(z) = (-1)^{l+1}\left(\frac{\pi}{2z}\right)^{1/2} J_{-l-1/2}(z) \tag{C.3}$$

which are *irregular* solutions of eq. (C.1)

iii) The *spherical Hankel functions* of the first and second kind

$$h_l^{(1)}(z) = j_l(z) + i n_l(z) \tag{C.4}$$

and

$$h_l^{(2)}(z) = j_l(z) - i n_l(z) \tag{C.5}$$

which are *irregular* solutions of eq. (C.1). Thus

$$j_l(z) = \tfrac{1}{2}[h_l^{(1)}(z) + h_l^{(2)}(z)] \tag{C.6}$$

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and

$$n_l(z) = \frac{1}{2i}[h_l^{(1)}(z) - h_l^{(2)}(z)]. \quad \text{(C.7)}$$

The pairs $\{j_l(z), n_l(z)\}$ and $\{h_l^{(1)}(z), h_l^{(2)}(z)\}$ are linearly independent solutions of eq. (C.1) for every l.

The first three functions j_l and n_l are given explicitly by

$$\begin{aligned} j_0(z) &= \frac{\sin z}{z}, \\ j_1(z) &= \frac{\sin z}{z^2} - \frac{\cos z}{z}, \\ j_2(z) &= \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \end{aligned} \quad \text{(C.8)}$$

and

$$\begin{aligned} n_0(z) &= -\frac{\cos z}{z}, \\ n_1(z) &= -\frac{\cos z}{z^2} - \frac{\sin z}{z}, \\ n_2(z) &= -\left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z. \end{aligned} \quad \text{(C.9)}$$

The functions $j_0(x)$, $j_1(x)$, $j_2(x)$ and $n_0(x)$, $n_1(x)$, $n_2(x)$, where x is real, are plotted in Figs. C.1 and C.2.

The functions $j_l(z)$ and $n_l(z)$ may be represented by the ascending series [1–3]

$$j_l(z) = \frac{z^l}{(2l+1)!!}\left[1 - \frac{\frac{1}{2}z^2}{1!(2l+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2l+3)(2l+5)} - \cdots\right] \quad \text{(C.10a)}$$

$$n_l(z) = -\frac{(2l-1)!!}{z^{l+1}}\left[1 - \frac{\frac{1}{2}z^2}{1!(1-2l)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2l)(3-2l)} - \cdots\right] \quad \text{(C.10b)}$$

where

$$(2l+1)!! = 1\cdot3\cdot5\cdots(2l+1).$$

We see from eqs. (C.10) that for $z \to 0$ one has

$$z^{-l}j_l(z) \underset{z\to 0}{\to} \frac{1}{(2l+1)!!}; \qquad j_l(z) \underset{z\to 0}{\sim} \frac{z^l}{(2l+1)!!}, \quad \text{(C.11a)}$$

$$z^{l+1}n_l(z) \underset{z\to 0}{\to} -(2l-1)!!; \qquad n_l(z) \underset{z\to 0}{\sim} -\frac{(2l-1)!!}{z^{l+1}}. \quad \text{(C.11b)}$$

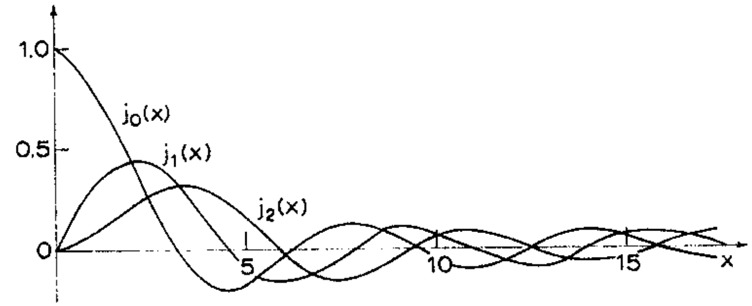

Fig. C.1. The first three spherical Bessel functions.

For real x somewhat larger than $l(l+1)$ one may use the asymptotic formulae

$$j_l(x) \underset{x\to\infty}{\to} \frac{1}{x}\sin(x - \tfrac{1}{2}l\pi), \quad \text{(C.12a)}$$

$$n_l(x) \underset{x\to\infty}{\to} -\frac{1}{x}\cos(x - \tfrac{1}{2}l\pi), \quad \text{(C.12b)}$$

$$h_l^{(1)}(x) \underset{x\to\infty}{\to} -i\,\frac{\exp\{i(x - \frac{1}{2}l\pi)\}}{x}, \quad \text{(C.12c)}$$

$$h_l^{(2)}(x) \underset{x\to\infty}{\to} i\,\frac{\exp\{-i(x - \frac{1}{2}l\pi)\}}{x}. \quad \text{(C.12d)}$$

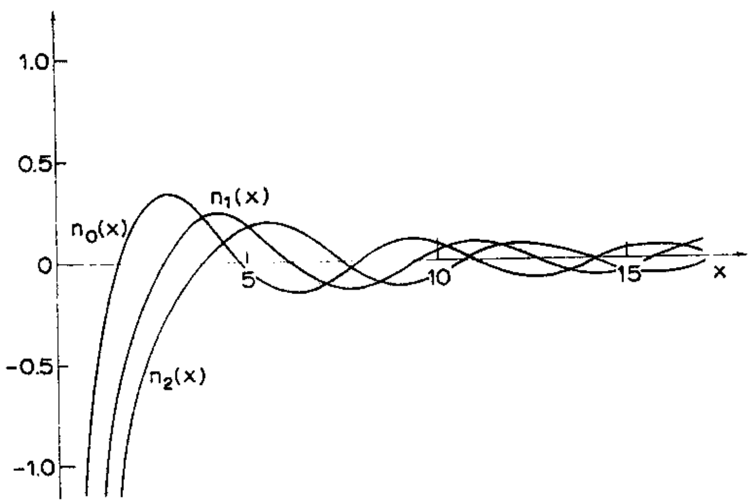

Fig. C.2. The first three spherical Neumann functions.

Some important properties of the spherical Bessel functions f_l [with $f_l : j_l, n_l, h_l^{(1)}, h_l^{(2)}$] are the recurrence relations (we assume that $l > 0$)

$$f_{l-1}(z) + f_{l+1}(z) = \frac{2l+1}{z} f_l(z), \tag{C.13a}$$

$$\frac{d}{dz} f_l(z) = \frac{1}{2l+1}[l f_{l-1}(z) - (l+1) f_{l+1}(z)] \tag{C.13b}$$

$$f_{l-1}(z) = \frac{l+1}{z} f_l(z) + \frac{d}{dz} f_l(z) \tag{C.13c}$$

$$f_{l+1}(z) = \frac{l}{z} f_l(z) - \frac{d}{dz} f_l(z). \tag{C.13d}$$

One also has the differentiation formulae

$$\frac{d}{dz}[z^{l+1} f_l(z)] = z^{l+1} f_{l-1}(z), \tag{C.14a}$$

$$\frac{d}{dz}[z^{-l} f_l(z)] = -z^{-l} f_{l+1}(z) \tag{C.14b}$$

and the analytic continuations (with $l, m = 0, 1, 2, \ldots$)

$$j_l(z\,e^{m\pi i}) = e^{ml\pi i} j_l(z), \tag{C.15a}$$

$$n_l(z\,e^{m\pi i}) = (-1)^m e^{ml\pi i} n_l(z), \tag{C.15b}$$

$$h_l^{(1)}(z\,e^{(2m+1)\pi i}) = (-1)^l h_l^{(2)}(z), \tag{C.15c}$$

$$h_l^{(2)}(z\,e^{(2m+1)\pi i}) = (-1)^l h_l^{(1)}(z), \tag{C.15d}$$

$$h_l^{(k)}(z\,e^{2m\pi i}) = h_l^{(k)}(z), \qquad k = 1, 2. \tag{C.15e}$$

In particular, we see that

$$j_l(-z) = (-1)^l j_l(z), \tag{C.16a}$$

$$n_l(-z) = (-1)^{l+1} n_l(z), \tag{C.16b}$$

$$h_l^{(1)}(-z) = (-1)^l h_l^{(2)}(z), \tag{C.16c}$$

$$h_l^{(2)}(-z) = (-1)^l h_l^{(1)}(z). \tag{C.16d}$$

Additional useful properties of the functions j_l and n_l are

$$j_l(z)\,n_{l-1}(z) - j_{l-1}(z)\,n_l(z) = z^{-2}, \qquad l > 0 \tag{C.17a}$$

$$j_l(z)\frac{d}{dz} n_l(z) - n_l(z)\frac{d}{dz} j_l(z) = z^{-2} \tag{C.17b}$$

$$\int j_0^2(x)\,x^2\,dx = \tfrac{1}{2}x^3[j_0^2(x) + n_0(x)\,j_1(x)] \tag{C.17c}$$

$$\int n_0^2(x)\,x^2\,dx = \tfrac{1}{2}x^3[n_0^2(x) - j_0(x)\,n_1(x)] \tag{C.17d}$$

$$\int j_1(x)\,dx = -j_0(x) \tag{C.17e}$$

$$\int j_0(x)\,x^2\,dx = x^2 j_1(x) \tag{C.17f}$$

$$\int j_l^2(x)\,x^2\,dx = \tfrac{1}{2}x^3[j_l^2(x) - j_{l-1}(x)\,j_{l+1}(x)], \qquad l > 0. \tag{C.17g}$$

The last three formulae are equally valid with the j's replaced by the corresponding n's.

Let us also quote a few definite integrals [4] involving the functions j_l and which are often used in scattering theory calculations, namely

$$\int_0^\infty e^{-ax} j_l(bx) x^{\mu-1}\,dx = \frac{\sqrt{\pi} b^l \Gamma(\mu + l)}{2^{l+1} a^{\mu+l} \Gamma(l + \frac{3}{2})} \times {}_2F_1\left(\frac{\mu+l}{2}, \frac{\mu+l+1}{2}; l + \tfrac{3}{2}; \frac{-b^2}{a^2}\right)$$

$$(\mathrm{Re}(a + ib) > 0,\ \mathrm{Re}(a - ib) > 0,\ \mathrm{Re}(\mu + l) > 0), \tag{C.18a}$$

$$\int_0^\infty e^{-ax} j_l(bx)\,x^{l+1}\,dx = \frac{(2b)^l\,\Gamma(l+1)}{(a^2+b^2)^{l+1}} \qquad (\mathrm{Re}\,a > |\mathrm{Im}\,b|), \tag{C.18b}$$

$$\int_0^\infty e^{-ax} j_l(bx)\,x^{l+2}\,dx = \frac{2a\,(2b)^l\,\Gamma(l+2)}{(a^2+b^2)^{l+2}} \qquad (\mathrm{Re}\,a > |\mathrm{Im}\,b|). \tag{C.18c}$$

Similar integrals involving higher powers of x may be obtained by differentiating with respect to the quantity a.

Finally, we note that

$$\int_0^\infty j_l(kr)\,j_l(k'r)\,r^2\,dr = \frac{\pi}{2k^2}\,\delta(k - k'). \tag{C.19}$$

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