Lecture 8

Where did we arrive at last time?

→ $SU(3)_c$ is motivated by the attempt to explain hadronic spectrum in term of fundamental fermions → quarks

→ works out only if assuming a new quantum number, color

How can we test if this theory is correct?

1. Compare $e^+e^- \to \mu^+\mu^-$ to $e^+e^- \to$ hadrons

$$\sigma_0^{\mu} = \frac{4\pi \alpha_{em}}{3s}$$

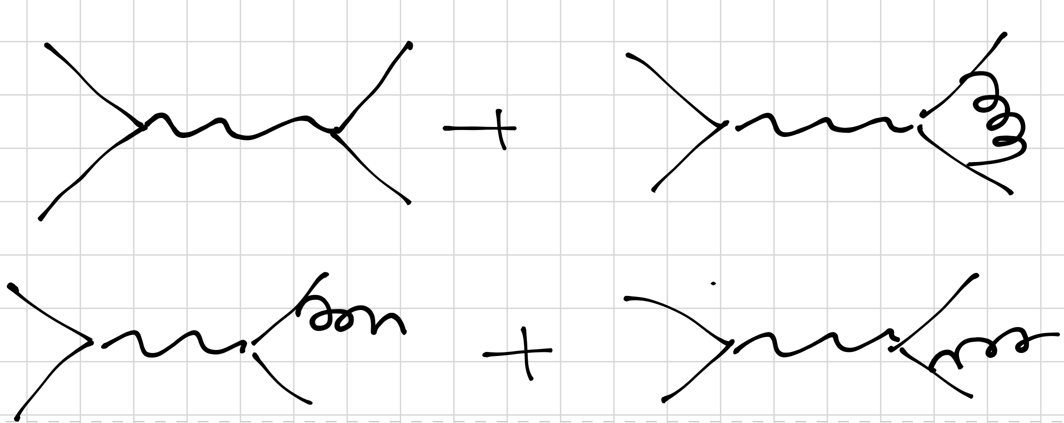
$$\sigma_0^{h(q\bar{q})} = 3\,\sigma_0^{\mu} \sum_f Q_f^2$$

3 → # of colors

$Q_f = \frac{2}{3}; -\frac{1}{3}$ quark charges.

Comparison for this prediction done at LEP 1 (CERN) at $s = M_Z^2$ consistent with $N_c = 3$ (to ~ 3,5 %)

To do better → include strong-interaction corrections



and

Calculation gives

$$R_{had} = \frac{\sigma^{had}}{\sigma^{\mu}} = R^{0}_{had}\left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$$

$$\alpha_s = \frac{g^2}{4\pi} \;; \quad \text{the prefactor is } \frac{3}{4} C_F = 1$$

Check Schwartz Chapter 26.3, or Peskin-Schroeder, Final problem to Part I for details of the calculation.

3.5% correction at Z-pole

$$\longrightarrow \alpha_s = 0.035 \cdot \pi = 0.11 \approx \frac{1}{9}$$

Compare to $\alpha_{em}(M_Z) \cong \frac{1}{129}$

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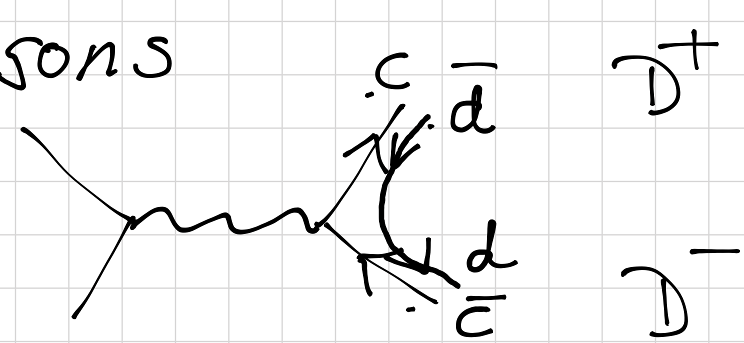
pQCD corrections are very often go as

$$1 + \frac{\alpha_s}{\pi} + \dots$$

Computing them to order α_s^4 or α_s^5 is an important skill (several groups)

Now, the correct scaling of $\sigma_{e^+e^- \to \text{hadrons}}$ is great but we do not observe quarks in the detector but hadrons

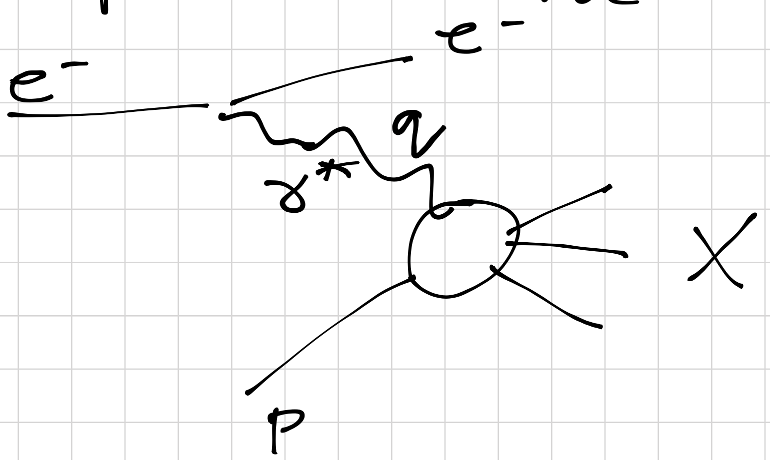
e.g. mesons



Can this scaling be explained in another way?

There is one more piece of information: cross section goes as $\frac{1}{s}$, that is quarks are as structure-less as muon. Other processes where this becomes evident?

Deep-Inelastic Scattering (DIS)



$q^2 = -Q^2 \qquad Q^2 \gg$

$W^2 = (p+q)^2 \gg$

mass of X

Infinite momentum frame: $P^\mu \approx (P^+, 0, \vec{0}_\perp)$

mass $\to 0$ $\qquad \bar{n} = \frac{1}{\sqrt{2}}(1,0,0,-1) \qquad (a\bar{n}) = \frac{1}{\sqrt{2}}(a^0 + a_3)$

$n = \frac{1}{\sqrt{2}}(1,0,0,1) \qquad (an) = \frac{1}{\sqrt{2}}(a^0 - a^3)$

$a^\mu = ((an)\cdot\bar{n} + (a\bar{n})\,n + \vec{a}_\perp) \qquad n\bar{n} = 1$

$(ab) = a^+ b^- + a^- b^+ - \vec{a}_\perp \vec{b}_\perp \qquad (an) = a^+$

$a^2 = 2a^+ b^- - \vec{a}_\perp^{\,2} \qquad (a\bar{n}) = a^-$

$P^\mu = (P^+, 0, \vec{0}_\perp) \qquad P^+ \gg$

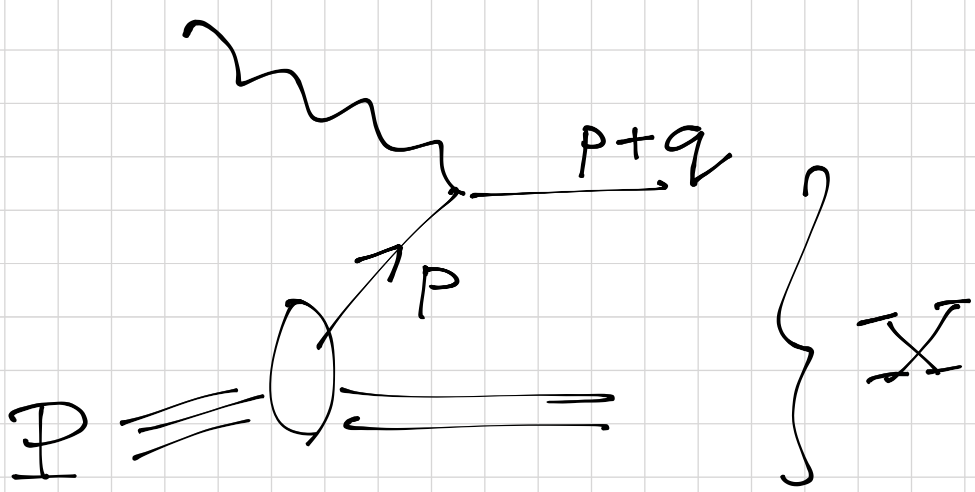
$q^\mu = \left(-x_B P^+, \frac{Q^2}{2x_B P^+}, 0_\perp\right) \qquad x_B \equiv \frac{Q^2}{2Pq}$

x_B – Bjorken variable

$Q^2 \gg$ and $W^2 \gg$ but x_B finite

$$W^2 = (p+q)^2 = Q^2\left(1 + \frac{1}{x_B}\right)$$

If the proton consists of structureless quarks, then one expects that in this kinematics electron effectively scatters off a quasi-free quark



Quasi-free: scattered quark is not bound

$$(p+q)^2 \sim m_q^2 = 0$$

What do we learn?

For $P^+ \gg$ $\quad p^+ = x P^+$, while other components are small ($p^2 \sim m_q^2 \approx 0$)

$$\rightarrow \quad p+q = \left((x - x_B)P^+, \frac{Q^2}{2x_B P^+}, \vec{p}_\perp\right)$$

$$(p+q)^2 = \frac{x - x_B}{x_B} Q^2 - \vec{p}_\perp^{\,2} \approx 0$$

$$\Rightarrow \boxed{x = x_B}$$

But x_B is entirely determined by exernal kinematics (E', Q^2)

Cross section for elastic $\bar{e}q$ scattering

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha_{em}^2}{\hat{s}^2} Q_f^2 \left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right)$$

$\hat{s}, \hat{t}, \hat{u} \to$ Mandelstam variables for e-q scattering, $\hat{s}+\hat{t}+\hat{u}=0$

$\hat{s} = (p+k)^2 = 2pk = x \cdot (P+k)^2 = x \cdot s$

$\hat{u} = -\hat{s} - \hat{t} = Q^2 - xs$

$\hat{t} = -Q^2$

$$\hookrightarrow \frac{d\sigma}{dt} = \frac{2\pi\alpha_{em}^2}{Q^4} Q_f^2 \left[1 + \left(1 - \frac{Q^2}{xs}\right)^2\right]$$

The quark momentum has some distribution function inside the proton $q(x)$

$\Downarrow$

$$\frac{d\sigma}{dx\,dQ^2} = \sum_f q_f(x)\, Q_f^2 \cdot \frac{2\pi\alpha^2}{Q^4}\left[1 + \left(1 - \frac{Q^2}{xs}\right)^2\right]$$

If factoring out the kinematics $\longrightarrow$

the cross section is independent of Q^2, but only depends on x.

<u>Bjorken scaling</u> $\longleftrightarrow$ quarks are (asymptotically) free and structureless

The first-ever DIS experiment at SLAC in 1973 observed exactly this picture

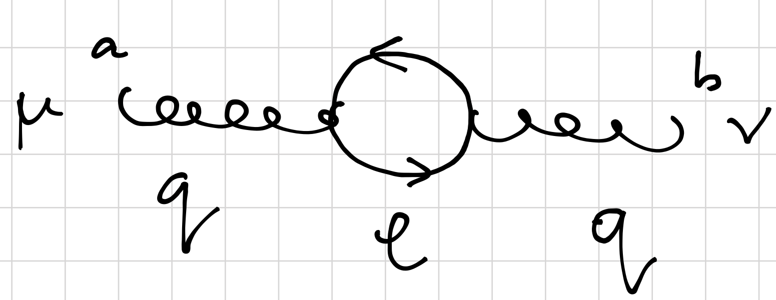
At high Q^2 quarks are $\approx$ free!
But at low Q^2 they are bound inside p

This is asymptotic freedom and confinement, central features of QCD!

If QCD as field theory is to be true, it should show this pattern.

To observe this pattern compute the gluon self-energy

Running α_s



$= \Pi_F^{ab\,\mu\nu}$

$$-g^2\,\mathrm{Tr}(T^aT^b)\int\frac{d^4\ell}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma^\mu(\not{\ell}-\not{q}+m)\gamma^\nu(\not{\ell}+m)]}{[(q-\ell)^2-m^2+i\varepsilon][\ell^2-m^2+i\varepsilon]}$$

$$=-g^2\,(g^{\mu\nu}q^2-q^\mu q^\nu)\,\delta^{ab}\,\Pi_2(q^2)$$

$$\to \mathrm{Tr}(T^aT^b)=T_F\delta^{ab}=\tfrac{1}{2}\delta^{ab}$$

$$\frac{1}{\ell^2-m^2+i\varepsilon}\;\frac{1}{\ell^2-2q\ell+q^2-m^2+i\varepsilon}$$

$$=\int_0^1\frac{dx}{[\underbrace{(\ell-xq)}_{\tilde{\ell}}{}^2-m^2+x(1-x)q^2+i\varepsilon]^2}$$

$$\mathrm{Tr}[\gamma^\mu(\tilde{\not{\ell}}-(1-x)\not{q}+m)\gamma^\nu(\tilde{\not{\ell}}+x\not{q}+m)]$$

$$=4\left[2\tilde{\ell}^\mu\tilde{\ell}^\nu+(m^2-\tilde{\ell}^2)g^{\mu\nu}-\underline{2x(1-x)q^\mu q^\nu}+x(1-x)q^2g^{\mu\nu}\right]$$

Odd powers of $\tilde{\ell}\to 0$ $\quad q^\mu\ell^\nu,\ (\ell q)\to 0$

$$\int d^4\ell\,\ell^\mu\ell^\nu f(\ell^2)=\frac{1}{4}g^{\mu\nu}\int d^4\ell\,\ell^2 f(\ell^2)$$

No need to compute everything!

Only one term $\sim q^\mu q^\nu$, the rest is $\sim q^2 g^{\mu\nu}$

$\Pi_2(q^2)\cdot(q^2g^{\mu\nu}-q^\mu q^\nu)$ by gauge inv.

$$\Rightarrow \Pi_2(q^2) = 4T_F \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx\, (+2x(1-x))$$

$$\cdot \frac{1}{\left[\ell^2 - m^2 + x(1-x)q^2 + i\varepsilon\right]^2}$$

Dim.-Reg.

$$\mu^\varepsilon \int \frac{d^{4-\varepsilon}\ell}{(2\pi)^{4-\varepsilon}} \frac{1}{[\ell^2 - \Delta]^2} = \mu^\varepsilon \frac{i\pi^{d/2}}{(2\pi)^d} \Delta^{-\varepsilon/2} \Gamma\left(\frac{\varepsilon}{2}\right)$$

$$= \frac{i\pi^2}{(2\pi)^4}\left[\frac{2}{\varepsilon} - \gamma_E - \ln\frac{\Delta}{\mu^2} + O(\varepsilon)\right]$$

$\mu \to$ renormalization scheme

$\Downarrow$

$$M_F^{ab\,\mu\nu} = -g^2 \delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot 4T_F \int_0^1 dx\, (+2x(1-x))$$

$$\cdot \frac{i\pi^2}{(2\pi)^4}\left[\frac{2}{\varepsilon} - \gamma_E + \ln\frac{\mu^2}{m^2 - q^2 x(1-x)}\right]$$

$$= -\frac{ig^2}{2\pi^2} T_F \delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \qquad m \to 0$$

$$\int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} - \gamma_E + \ln\left(\frac{\mu^2}{-q^2 + i\varepsilon}\right) - \ln x(1-x)\right]$$

a). $\int_0^1 dx\, x(1-x) = 1/6$

b). $\int_0^1 dx\, x(1-x) \ln x(1-x) = -\frac{5}{18}$

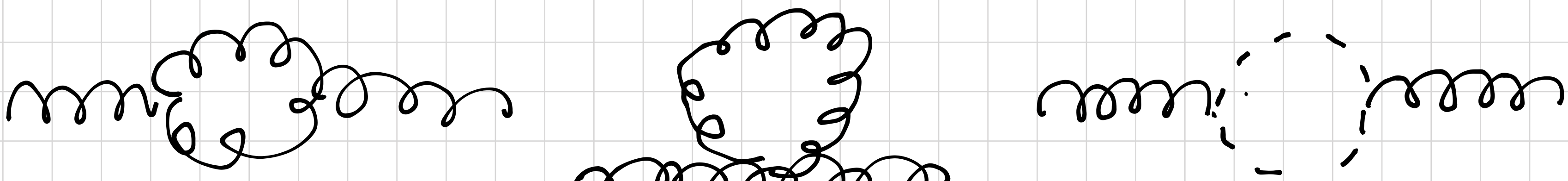
$\Downarrow$ $\qquad\qquad$ $\alpha_s = \frac{g^2}{4\pi}$

$$-iM_F^{ab\,\mu\nu} = \frac{\alpha_s}{4\pi} T_F \delta^{ab} (q^2 g^{\mu\nu} - q^\mu q^\nu)$$
$$\cdot \left\{ -\frac{4}{3}\left(\frac{2}{\varepsilon} - \gamma_E + \ln \frac{\mu^2}{-q^2+i\varepsilon}\right) - \frac{20}{9} \right\}$$

OK

Same as VP in QED w. $\alpha \to \alpha_s T_F \delta^{ab}$

The real news: gluonic and ghost loops



All crucial steps discussed in Schwartz, Ch. 26.4
$\longrightarrow$ just fill in details!

*) Important discussion on divergences in 3g and 4g graphs (beyond dim reg)

**) I used the trick in quark VP: only computed term $\sim -q^\mu q^\nu$ assuming the rest <u>has</u> to be $q^2 g^{\mu\nu}$
This is cheating in general case
(I just knew that it works in QED where there is just one diag.)
Need to do explicitly at least once in your life!

After you fought through the exercise:

You arrive at

$$\mathcal{M}^{ab\mu\nu} = \delta^{ab} \frac{\alpha_s}{4\pi} (q^2 g^{\mu\nu} - q^\mu q^\nu) \left[\frac{5}{3} C_A - \frac{4}{3} n_f T_F \right]$$
$$\times \left\{ \frac{2}{\varepsilon} - \gamma_E + \ln \frac{\mu^2}{-q^2 + i\varepsilon} \right\}$$

$T_F = \frac{1}{2}$

n_f — # of flavors in the loop

$\binom{u}{d} \binom{c}{s} \binom{t}{b} \longrightarrow n_f = 6$ in principle

In practice, we worked in the limit $m_q \ll |q^2| \Rightarrow$ for a given q^2 only $m_q^2 < |q^2|$ participate in running, heavy quarks decouple

$$\ln \frac{\mu^2}{M^2 - q^2 x(1-x)} \rightarrow \ln \frac{\mu^2}{m^2} = \text{const.}$$

$C_A = N_c = 3$

For QCD $\quad 5 - \frac{2}{3} n_f > 0$

For QED only the second term is present

$C_A \to 0, \quad n_f \to 2$

The sign of the β-function tells how the effective coupling behaves asymptotically.

Next discuss renormalization in $\overline{MS}$ scheme (modified minimally-subtracted)

$$\mathcal{L} = -\frac{1}{4} Z_3 (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2$$

$$+ Z_2 \bar{\psi}_i (i\not{\partial} - Z_m m_R) \psi_i \quad - Z_{3c} \bar{c}^a \Box c^a$$

$$- g_R Z_{A^3} f^{abc} (\partial_\mu A_\nu^a) A^{b\mu} A^{c\nu}$$

$$- \frac{1}{4} g_R^2 Z_{A^4} (f^{eab} A_\mu^a A_\nu^b)(f^{ecd} A^{c\mu} A^{d\nu})$$

$$+ g_R Z_1 A_\mu^a \bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j$$

$$+ g_R Z_{1c} f^{abc} (\partial_\mu \bar{c}^a) A^{b\mu} c^c$$