Lecture 6

Gluon propagator, Faddeev – Popov ghosts

EoM of a photon $(\Box g^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu = J_\nu$

Cannot solve directly: $k^\mu k^\nu - k^2 g^{\mu\nu}$ has a 0 - mode (k_μ w. eigenvalue 0)

$\longrightarrow$ include a Lagrange multiplier $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to break the degeneracy

More systematically done with Path Integral

$$f(\xi) = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2} = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2}$$

is independent of A^μ (always can shift $\pi \to \pi - \frac{1}{\Box}\partial_\mu A^\mu$)

$$\hookrightarrow \int DA^\mu D\varphi_i \, e^{i\int d^4x \, \mathcal{L}}$$

$$= \frac{1}{f(\xi)} \int D\pi \, DA^\mu D\varphi_i \, e^{i\int d^4x \left(\mathcal{L} - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2\right)}$$

$$= \left[\frac{1}{f(\xi)} \int D\pi\right] \int DA^\mu D\varphi_i \, e^{i\int d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]}$$

Correlation fn. independent of gauge fixing

We wish to do the same for the non-abelian case

One auxiliary field π (abelian) $\rightarrow$ for non-abelian $\rightarrow (N^2-1)$ fields π^a (adj. represent.)

$$A^a_\mu \rightarrow A^a_\mu + \frac{1}{g}\partial_\mu \pi^a + f^{abc} A^b_\mu \pi^c$$
$$= A^a_\mu + \frac{1}{g} D_\mu \pi^a \quad ; \quad D_\mu \pi^a = \partial_\mu \pi^a + g f^{abc} A^b_\mu \pi^c$$

Introduce a <u>functional of A</u>

$$f[A] = \int D\pi \exp\left[-i \int d^4x \frac{1}{2\xi}\left(\partial^\mu D_\mu \pi^a\right)^2\right]$$
$\uparrow$

Because the gauge transf. contains A in the non-abelian case

Shift $f[A] = \int D\pi \exp\left[-i \int d^4x \frac{1}{2\xi}\left(\partial_\mu A^{\mu a} - \partial_\mu D^\mu \pi^a\right)^2\right]$

should not do anything to the $\int$

$$\int DA_\mu D\varphi_i\, e^{i\int d^4x \mathcal{L}} = \int D\pi\, DA_\mu\, D\varphi_i \frac{1}{f[A]}$$
$$\times \exp\left[i \int d^4x \left(\mathcal{L} - \frac{1}{2\xi}\left(\partial_\mu A^{\mu a} - \partial_\mu D^\mu \pi^a\right)^2\right)\right]$$
$$= \left[\int D\pi\right] \int DA_\mu D\varphi_i \frac{1}{f[A]} e^{i\int d^4x\left[\mathcal{L} - \frac{1}{2\xi}\left(\partial_\mu A^{\mu a}\right)^2\right]}$$

upon $A^a_\mu \rightarrow A^a_\mu + D_\mu \pi^a$ under which $\mathcal{L}$ is <u>inv.</u>

Again, $\int D\pi$ can be taken out (as an

unphysical constant, likely ∞)
that will eventually drop out.
The difference with abelian case:
because of gluon self-interaction
$f[A]$ is a functional of gauge fields
and cannot be taken outside.

To proceed: use the definition of $f[A]$:

$$f[A] = \int D\pi \exp\left[-i \int d^4x \frac{1}{2\xi} \left(\partial_\mu D^\mu \pi^a\right)^2\right]$$

Recall that it is quadratic in $\pi \longrightarrow$ we can still perform the Gauss integral

$$\int d\vec{p}\ e^{-\frac{1}{2}\vec{p}A\vec{p}} = \frac{\text{const}}{\sqrt{\det A}}$$

$$\hookrightarrow f[A] = \frac{\text{const}}{\sqrt{\det (\partial_\mu D^\mu)^2}}$$

$\Downarrow$

$$Z[0] = \text{const.} \times \int DA^\mu D\varphi_i \left[\det(\partial_\mu D^\mu)\right]$$

$$\times \exp\left[i \int d^4x \left[\mathcal{L}[A,\varphi_i] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]\right]$$

We use the fact that due to properties of Grassmann numbers ($\int dx = \partial/\partial x$) in a similar Gauss integral for fermionic fields the det. appears in the numerator

$$\det(B) = \int D\bar{\psi}\, D\psi \, \exp\left(-i\int d^4x\, \bar{\psi} B \psi\right)$$

$$\hookrightarrow \det(\partial^\mu D_\mu) = \int D\bar{c}\, Dc \; e^{-i\int d^4x\, \bar{c}^a \partial_\mu D^\mu c^a}$$

For Grassmann-valued classical fields $c, \bar{c}$

Gauge-fixed path integral for non-abelian gauge theory:

$$Z[0] = \int DA^\mu D\varphi_i\, D\bar{c}\, Dc$$

$$\times \exp\left[i\int d^4x \left[\mathcal{L}[A,\varphi_i] - \frac{1}{2\xi}(\partial_\mu A^\mu_a)^2 - \bar{c}^a \partial_\mu D^\mu c^a \right]\right]$$

$c, \bar{c} \longrightarrow$ Faddeev-Popov ghosts
anticommuting Lorentz scalars

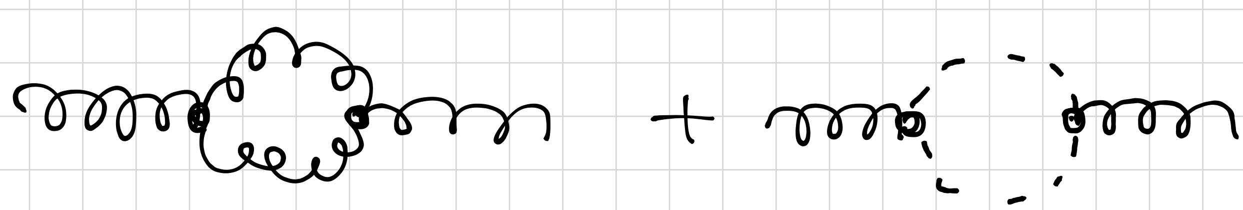
For each gauge field A^μ_a one ghost c^a and one anti-ghost $\bar{c}^a$

Where do they come from?

Lorentz covariant A^μ have 4 d.o.f.

But massless spin-1 field only has 2 transverse polarizations
We need to get rid of the two unphysical polarizations. In the abelian case they decouple trivially, but for non-abelian case because of self-interaction they don't! This is the price to pay for explicitly Lorentz-covariant form.
Ghosts are not physical $\leftrightarrow$ will not appear in observables (S-matrix) as external lines but can appear as internal lines in Feynman diags.

Ghosts are there to cancel unphys. polarizations of gluons in loops

e.g. 

+1 from boson loop

−1 from fermion loop

It turns out, the ghosts preserve a global symmetry $\rightarrow$ hence they have to be included

$$\mathcal{L}_{FP} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \left(D_\mu \varphi^*\right)\left(D^\mu \varphi\right) - m^2 \varphi^* \varphi$$

$$-\frac{1}{2\xi}\left(\partial_\mu A^{\mu a}\right)^2 + \left(\partial_\mu \bar{c}^a\right)\left(D^\mu c^a\right)$$

Consider transformations

$$\begin{cases} \varphi \to \varphi + i\Theta c^a T^a \varphi \\ A^a_\mu \to A^a_\mu + \frac{1}{g}\Theta D_\mu c^a \\ \bar{c}^a \to \bar{c}^a - \frac{1}{g}\Theta \frac{1}{\xi}\partial_\mu A^a_\mu \end{cases}$$

! c and $\bar{c}$ are not related

$$\hookrightarrow D_\mu c^a \to D_\mu c^a + \Theta f^{abc}(D_\mu c^b)c^c$$

With the additional shift

$$c^a \to c^a - \frac{1}{2}\Theta f^{abc} c^b c^c$$

$$D_\mu c^a \to D_\mu c^a + \Theta f^{abc}(D_\mu c^b) c^c$$
$$- \Theta f^{abc}\Big[\frac{1}{2}(\partial_\mu c^b) c^c + \frac{1}{2} c^b(\partial_\mu c^c)$$
$$+ \frac{g}{2} A^b_\mu f^{cde} c^d c^e\Big]$$

First the abelian case

$$-\frac{1}{4}F_{\mu\nu}^2 + (D_\mu \varphi^*)(D^\mu \varphi) - m^2 \varphi^*\varphi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\Box c$$

term $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ breaks gauge inv. down to a residual symmetry $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$ w. $\Box\alpha = 0$

Since $\Box c = \Box \bar{c} = 0$ from $\mathcal{L}$ eqs.

$\hookrightarrow$ use $\alpha = \Theta c$ for some $\Theta \to$ Grassmann #

$$\varphi \to \varphi + i\Theta c\, \varphi$$

$$A_\mu \to A_\mu + \frac{1}{e}\Theta \partial_\mu c$$

↳ these leave $-\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu \varphi^*)(D^\mu \varphi)$
$- m^2 \varphi^* \varphi$ invariant

$$(\partial_\mu A^\mu)^2 \longrightarrow (\partial_\mu A^\mu)^2 + \frac{2}{e}(\partial_\mu A^\mu)(\theta \Box c) + \underbrace{\frac{1}{e^2}(\theta \Box c)(\theta \Box c)}_{\sim \theta^2 = 0}$$

$\downarrow$

$$\bar{c} \to \bar{c} - \frac{1}{e}\theta \frac{1}{\xi}\partial_\mu A^\mu$$

$$\hookrightarrow -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - (\bar{c}\Box c) \longrightarrow -\frac{1}{2}(\partial_\mu A^\mu)^2 - \bar{c}\Box c \cancel{- \frac{1}{e\xi}(\partial_\mu A^\mu)(\theta \Box c)} \cancel{+ \frac{1}{e\xi}(\partial_\mu A^\mu)(\theta \Box c)}$$

$$\hookrightarrow \mathcal{L} = \mathcal{L}[A, \varphi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\Box c$$

has a global invariance under

$$\varphi \to \varphi + i\theta c \varphi \qquad \left(e^{i\theta c(x)} \varphi \right)$$
$$A_\mu \to A_\mu + \frac{1}{e}\theta \partial_\mu c$$
$$\bar{c} \to \bar{c} - \frac{1}{e}\theta \frac{1}{\xi}\partial_\mu A^\mu$$

Now non-abelian

$$\mathcal{L}_{FP} = \mathcal{L}[A^a_\mu, \varphi] - \frac{1}{2\xi}(\partial_\mu A^\mu_a)^2 + (\partial_\mu \bar{c}^a)(D^\mu c^a)$$

$$\left.\begin{array}{l} \varphi \to \varphi + i\theta c^a T^a \varphi \\ A^a_\mu \to A^a_\mu + \frac{1}{g}\theta D_\mu c^a \end{array}\right\} \to \text{leave } \mathcal{L}[A^a_\mu, \varphi] \text{ invariant}$$

$$\bar{c}^a \to \bar{c}^a - \frac{1}{g}\theta\frac{1}{\xi}(\partial_\mu A^{\mu a})$$

↳ designed to cancel the cross-term from $(\partial_\mu A^\mu)^2$

⊛ [Note $\partial_\mu \bar{c}^a D^\mu c^a \leftrightarrow -\bar{c}\Box c$ in abelian case: $\partial_\mu \bar{c}\,\partial^\mu c = -\bar{c}\Box c$ when integrating by parts in $\int d^4x\, \mathcal{L}$]

But now $D_\mu c^a \longrightarrow D_\mu c^a + \theta f^{abc}(D_\mu c^b) c^c$

↳ need to also transform

$$c^a \longrightarrow c^a - \frac{1}{2}\theta f^{abc} c^b c^c$$

$$D_\mu c^a \to D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c$$
$$-\frac{1}{2}\theta f^{abc}\Big[\underbrace{(\partial_\mu c^b)c^c + c^b(\partial_\mu c^c)}_{\text{are equal}} + gA_\mu^b f^{cde} c^d c^e\Big]$$

are equal $\left(\begin{array}{l} c^b \leftrightarrow c^c \text{ "}-\text{"} \\ f^{abc} = -f^{acb} \end{array}\right)$

$$= D_\mu c^a + \theta f^{abc}\Big[\cancel{(\partial_\mu c^b) c^c} + g f^{bde} A_\mu^d c^e c^c$$
$$\cancel{-(\partial_\mu c^b) c^c} - \frac{1}{2} g A_\mu^b f^{cde} c^d c^e$$

To cancel the second term: use Jakobi ID:

$$f^{abc} f^{cde} + f^{bdc} f^{cae} + f^{dac} f^{cbe} = 0$$

$$\Rightarrow f^{abc} f^{bde} A_\mu^d c^e c^c + \frac{1}{2} f^{bdc} f^{cae} A_\mu^b c^d c^e$$
$$+ \frac{1}{2} f^{dac} f^{cbe} A_\mu^b c^d c^e$$

$$
= f^{abc} f^{bde} A_\mu^d c^e c^c + \underbrace{\tfrac{1}{2} f^{dbc} f^{cae} A_\mu^d c^b c^e}_{\begin{array}{l} = \frac{1}{2} f^{bde} f^{bac} A_\mu^d c^e c^c \\ = -\frac{1}{2} f^{abc} f^{bde} A_\mu^d c^e c^c \end{array}}
$$

$$
+ \underbrace{\tfrac{1}{2} f^{cde} f^{bac} A_\mu^d c^b c^e}_{\begin{array}{l} = \frac{1}{2} f^{bde} f^{cab} A_\mu^d c^c c^e \\ = \frac{1}{2} f^{abc} f^{bde} A_\mu^d c^c c^e \end{array}}
$$

$$
= \emptyset
$$

$$
\Longrightarrow \underline{\mathcal{L} \longrightarrow \mathcal{L}}
$$

This symmetry is called BRST

<u>B</u>ecchi – <u>R</u>ouet – <u>S</u>tora – <u>T</u>yutin.

As we saw, it is also a symmetry of the QED Lagrangian;

But there any gauge-fixing term couples to a <u>conserved</u> current $\rightarrow$ ghosts decouple

BRST symmetry crucial for proving renormalizability of QCD.

Gauge part of the Lagrangian
(gauge-fixed)

$$\mathcal{L}_{R_\xi} = -\frac{1}{4}\left(F_{\mu\nu}^{a}\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A_a^{\mu}\right)^2$$
$$+ \left(\partial_\mu \bar{c}^a\right)\left(\delta^{ab}\partial^\mu + g f^{acb} A^{c\mu}\right) c^b$$

Can read off the gluon propagator just like that of the photon modulo color indices

$$= i \, \frac{-g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)\frac{q^\mu q^\nu}{q^2}}{q^2 + i\varepsilon}\, \delta^{ab}$$

The only way to get rid of ghosts $\rightarrow$ give up Lorentz invariance

Example: axial gauges
("axis" $\rightarrow$ selected vector
$\rightarrow$ pre-selected direction in space)

$$\mathcal{L}_{gauge-fix} + \mathcal{L}_{ghosts} = -\frac{1}{2\lambda}\left(r^\mu A_\mu^a\right)^2$$

$$+\bar{c}^{a} r^{\mu} D_{\mu} c^{c}$$

$r^{\mu} \to$ chosen vector ; $\lambda \to$ parameter

Gluon propagator:

$$i\Pi^{\mu\nu} = i\,\frac{-g^{\mu\nu} + \frac{r^{\mu}q^{\nu} + r^{\nu}q^{\mu}}{(rq)} - \frac{(r^2+\lambda q^2)\,q^{\mu}q^{\nu}}{(rq)^2}}{q^2+i\varepsilon}\,\delta^{ab}$$

E.g. $r^{\mu} = (1,\vec{0}) \to -\frac{1}{2\lambda}(A^{0}_{a})^2 \quad \lambda\to 0 \Rightarrow A^{0}=0$

For $q^2=0$ (on-shell) $\quad q_{\mu}\, i\Pi^{\mu\nu} = 0$

$$r_{\mu}\, i\Pi^{\mu\nu} = i\,\frac{-\lambda q^2}{rq}\, q^{\nu} \underset{\lambda=0}{=} 0$$

Since ghost-antighost-gluon vertex $\sim r^{\mu}$

$\hookrightarrow$ ghosts decouple!

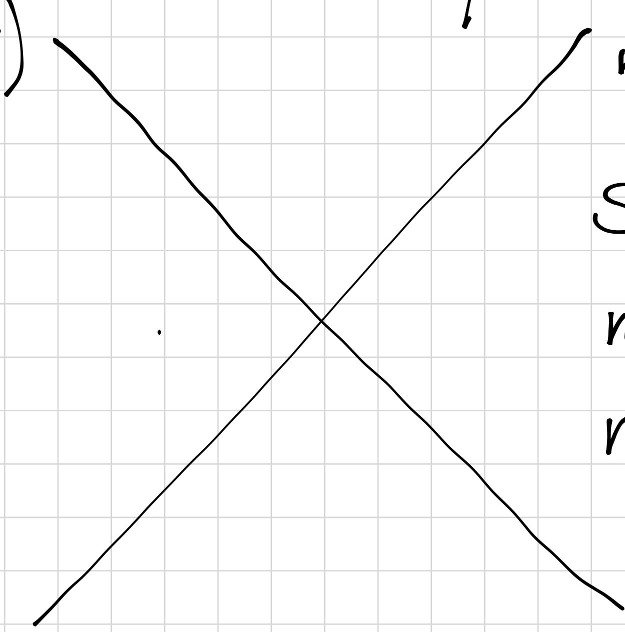
Special case: $r^2=0$ (light-like)

$\lambda = 0$

$\hookrightarrow r^{\mu}A^{a}_{\mu} = 0$

$$i\Pi^{\mu\nu,ab} = i\,\frac{-g^{\mu\nu} + \frac{r^{\mu}q^{\nu}+r^{\nu}q^{\mu}}{(rq)}}{q^2+i\varepsilon}\,\delta^{ab}$$

$\bar{n} = \frac{1}{\sqrt{2}}(1,0,0,-1)$ $\quad$ $n = \frac{1}{\sqrt{2}}(1,0,0,1)$



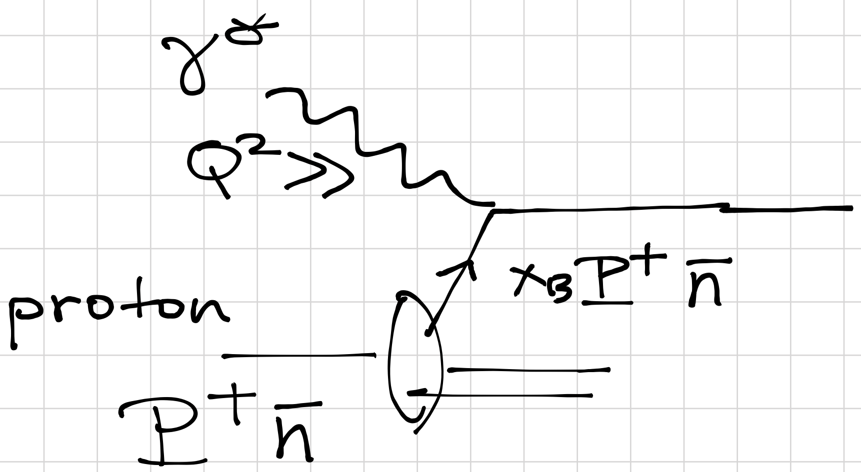
Sudakov vectors

$n^2 = \bar{n}^2 = 0$

$n\bar{n} = 1$

Often in perturbative QCD calcula-
tions lightcone gauge is useful

Asymptotic energy → infinite momentum frame (all masses can be put to 0)



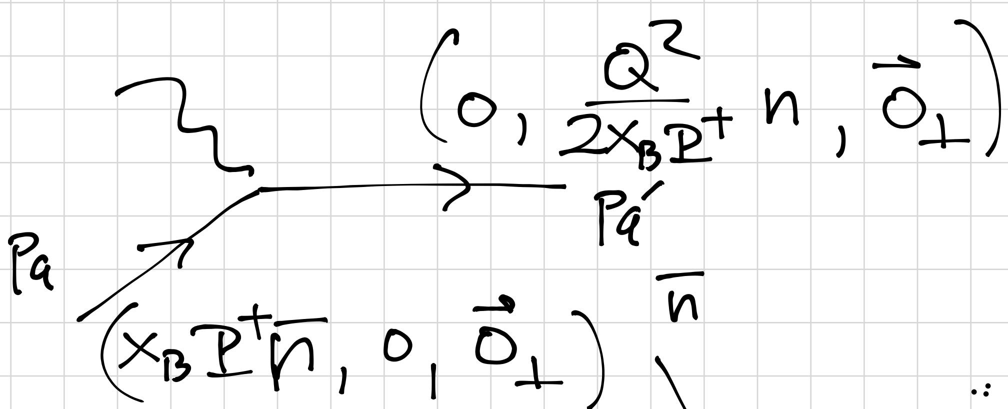
$$q^\mu = (\alpha n + \beta \bar{n}, \vec{0}_\perp)$$

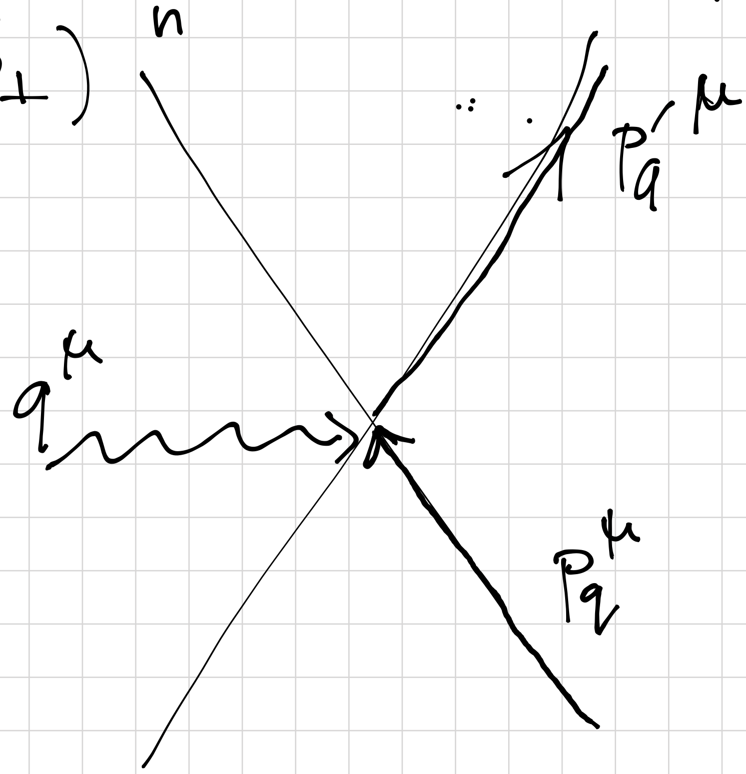
$$q^2 = -Q^2 = 2\alpha\beta$$

$$2pq = 2\alpha P^+ = \frac{Q^2}{x_B}$$

$$\alpha = +\frac{Q^2}{2x_B P^+}$$

$$\beta = -x_B P^+$$





Then, if the intermediate quark emits a gluon ~ $\bar{u}(p_{q'} - q_g)\gamma^\alpha u(p_{q'})$ vertex

only "+" component is large $\parallel \bar{n}$

"−" component ≈ 0

$\not{n}\, u(p_{q'}) = \# \not{p}_{q'}\, u(p_{q'}) = \# \cdot m_q\, u(p_q) \approx 0$

$\hookrightarrow$ it makes sense to use lightcone gauge to only compute non-zero terms in the lightcone expansion