Lecture 5

Non-Abelian Gauge Field Theory

1. Free fermion theory

$$\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - \mu)\psi$$

Global symmetry $\psi \to e^{+i\alpha}\psi$; $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$

$\alpha = \text{const.}$

Just an overall phase

If now requiring a symmetry under a <u>local</u> phase transformation $\alpha = \alpha(x)$, the Lagrangian is not invariant:

$$\bar{\psi} i\not{\partial}\psi \to \bar{\psi} e^{-i\alpha(x)} i\not{\partial} e^{+i\alpha(x)}\psi$$

$$= \bar{\psi} i\not{\partial}\psi - \bar{\psi}\gamma_\mu\psi(\partial^\mu\alpha(x))$$

! Free theory cannot be invariant under local phase transformation

However, the invariance can be restored in presence of the U(1) gauge field A^μ

$$A^\mu \to A^\mu + \frac{1}{e}\partial^\mu\alpha(x)$$

$$\partial^\mu \to D^\mu = \partial^\mu - ieA^\mu$$

$$\psi \to e^{i\alpha(x)}\psi$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

is gauge invariant

$$F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\cancelto{0}{[A_\mu, A_\nu]}$$

This is the case for a single complex field ψ

Symmetry group $= U(1)$

For the case of N complex ψ

Local phase transformation will have a matrix representation

$$\vec{\psi} \to e^{i\theta^a T^a} \vec{\psi} \qquad \vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$$

θ^a : N^2-1 real numbers

T^a : $N \times N$ matrices, also N^2-1 species

$\mathrm{Tr}(T^a) = 0$; $T^a = (T^a)^\dagger$

$T^a \to$ generators of Lie group

T^a's form a Lie algebra:

$$[T^a, T^b] = i f^{abc} T^c$$

$f^{abc} \to$ structure constants

If $f = 0 \to$ abelian Lie group ($U(1)$, QED)

$f \neq 0 \to$ non-abelian ($SU(2) \to$ weak int.

$SU(3) \to$ QCD)

Standard Model is a non-abelian

gauge field theory $SU(3) \times SU(2) \times U(1)$

Choose the normalization of T^a such that

$$\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}$$

$SU(2)$ (special unitary)

special: $\det U = \det\left(e^{i\theta^a T^a}\right) = +1$

$$T^a = \tau^a = \frac{1}{2}\sigma^a \quad \text{Pauli matrices} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[T^a, T^b] = i\varepsilon^{abc}T^c \qquad f^{abc} = \varepsilon^{abc}$$

$SU(3)$ $\quad T^a = \frac{1}{2}\lambda^a \quad$ Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

Above are representations (basis) of respective Lie algebra

$$\sum_{c,d} f^{acd} f^{bcd} = N\,\delta^{ab}$$

f^{abc} totally antisymmetric (e.g. ε^{abc} Levi-Civitá)

Product of generators

$$T^a T^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2}d^{abc}T^c + \frac{i}{2}f^{abc}T^c$$

$$d^{abc} = 2\,\mathrm{Tr}\left[T^a\{T^b, T^c\}\right]$$

For $SU(2)$ $\quad d^{abc} = 0 \quad \{T^b, T^c\} = \frac{1}{2}\delta^{bc}, \quad \mathrm{Tr}(T^a) = 0$

$$\mathrm{Tr}\left[T^a T^b T^c\right] = \frac{1}{4}\left(d^{abc} + i f^{abc}\right)$$

$$\mathrm{tr}\left[T^a T^b T^c T^d\right] = \frac{1}{4N}\delta^{ab}\delta^{cd} + \frac{1}{8}\left(d^{abe} + i f^{abe}\right)\left(d^{cde} + i f^{cde}\right)$$

etc.

Adjoint representation: matrices acting in the a-dimensional space of Lie algebra generators

$SU(N)$: N^2-1 of $(N\times N)$ generators

$$\left(T^a_{adj}\right)^{bc} = -i f^{abc} \qquad \left[T^a_{adj}, T^b_{adj}\right] = i f^{abc} T^c_{adj}$$

$$\mathrm{Tr}\left[T^a_{adj} T^b_{adj}\right] = N$$

$SU(2)$:

$$T^1_{adj} = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} \quad T^2_{adj} = \begin{pmatrix} 0 & & i \\ & 0 & \\ -i & & 0 \end{pmatrix} \quad T^3_{adj} = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}$$

$SU(3)$: 8 matrices

Casimir op. $\quad \sum_a T^a_R T^a_R = C_2^R \mathbb{1}$

R: representation

C_2^R: quadratic Casimir for a given R

For $SU(N)$:

$$C_2^{R=F} = \frac{N^2-1}{2N} \equiv C_F$$

$R=F$: fundamental

$SU(2)$: $\frac{3}{4}$

$SU(3)$: $\frac{4}{3}$

Fierz ID: $\sum_a T^a_{ij} T^a_{kl} = \frac{1}{2}\left[\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right]$

$$\to \mathrm{Tr}[T^a A]\,\mathrm{Tr}[T^a B]$$

$$= \frac{1}{2}\left[\mathrm{Tr}(AB) - \frac{1}{N}\mathrm{Tr}A \cdot \mathrm{Tr}B\right]$$

With these ingredients generalize gauge inv. to non-abelian case:

Gauge transf. $\psi \to e^{i\theta^a T^a}\psi$, with $T^a \to (N\times N)$ matrix, $\psi \to N-$vector

$A_\mu(x) = A^a_\mu(x)\,T^a$ $N\times N$ matrix

$$D_\mu = \partial_\mu \cdot \mathbb{1}_N - ig A_\mu$$

Field strenght $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

Now $F_{\mu\nu} = F^a_{\mu\nu} T^a$, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$$

These manipulations make the kinetic term gauge-invariant

$\downarrow$

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{\psi}(i\not{D} - m)\psi$$

In SM quarks also come in 6 flavors

u - up   c - charm   t - top

d - down   s - strange   b - bottom

↳ gauge part of the Lagrangian is still the same

↳ Mass becomes a diag_6 matrix

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & \ddots \end{pmatrix}$$

$m_u \simeq 2\,\mathrm{MeV}$

$m_d \simeq 5\,\mathrm{MeV}$

$m_s \simeq 100\,\mathrm{MeV}$

$m_c \simeq 1300\,\mathrm{MeV}$

$m_b \simeq 4200\,\mathrm{MeV}$

$m_t \cong 173\,000\,\mathrm{MeV}$

Wilson lines

We transformed fields by a local phase

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \; ; \; \psi(y) \to e^{i\alpha(y)}\psi(y)$$

At point x the phase is merely a convention. In a local theory the choice at x and y should be independent → but if so, how is the derivative defined?

$$\phi(y) - \phi(x) \longrightarrow e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x)$$

Introduce a bi-local Wilson line

$$W(x,y) \longrightarrow e^{i\alpha(x)} W(x,y)\, e^{-i\alpha(y)}$$

$$W(x,y)\,\phi(y) - \phi(x) \longrightarrow e^{i\alpha(x)}\left(W(x,y)\,\phi(y) - \phi(x)\right)$$

$$D_\mu \varphi(x) = \left.\frac{W(x, x+\delta x)\,\phi(x+\delta x) - \phi(x)}{\delta x}\right|_{\delta x \to 0}$$

$$W(x,x) = 1 \hookrightarrow W(x, x+\delta x) = 1 - ie\,\delta x^\mu A_\mu + \ldots$$

$$\hookrightarrow A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

and $D_\mu = \partial_\mu - ieA_\mu$ is the cov. derivative

Closed form expression for W:

$$W(x,y) = \exp\left[ie\int_y^x A_\mu(z)\,dz^\mu\right]$$

Check transformation

$$W(x,y) \longrightarrow \exp\left[ie\int_y^x A_\mu(z)\,dz^\mu + i\int_y^x \partial_\mu \alpha(x)\,dz^\mu\right]$$

$$= e^{i\alpha(x)}\, W(x,y)\, e^{-i\alpha(y)} \qquad \underline{\text{OK}}$$

Wilson line is an $\int$ along the path

$$z^\mu(\lambda) \quad \text{for } \lambda \in [0,1]$$

$$z(0) = y, \quad z(1) = x$$

$$W_P(x,y) = \exp\left[ie \int_0^1 \frac{dz^\mu(\lambda)}{d\lambda} A_\mu(z(\lambda))\, d\lambda\right]$$

The phase transformation is P-indep.!
only depends on endpoints (x,y).

If $x=y \rightarrow$ Wilson loop – $\int$ over closed contour

$$W_P^{loop} = \exp\left(ie \oint_P A_\mu dx^\mu\right)$$

Loop is gauge-invariant
by Stoke's theorem

$$\oint_P A_\mu dx^\mu = \frac{1}{2}\int_\Sigma F_{\mu\nu}\, d\sigma^{\mu\nu}$$

$\leftarrow$ surface

$\hookrightarrow$ gauge-inv.

Can easily be extended to non-abelian case

$$A_\mu \longrightarrow A_\mu^a T^a$$

$$W_P(x,y) = \mathcal{P}\left\{\exp\left(ig\int_y^x A_\mu^a(z) T^a dz^\mu\right)\right\}$$

Path-ordering: group generators at different points do not commute

$$\hookrightarrow \quad W_P(x,y) = 1 + ig\int_0^1 \frac{dz^\mu}{dx} A_\mu^a T^a\, d\lambda$$

$$-\frac{1}{2}g^2\int_0^1 d\lambda \int_0^1 d\tau \, \frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\tau} A^a_\mu(z(\lambda))\, A^b_\nu(z(\tau))$$

$$\times\left[T^a T^b \Theta(\lambda-\tau) + T^b T^a \Theta(\tau-\lambda)\right] + \ldots$$

Finally, the locally $SU(N)$ invariant $\mathcal{L}$

$$\boxed{\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\Psi}\left(i\not{\partial}\cdot \mathbb{1} - M + g\not{A}^a T^a\right)\Psi}$$

One more gauge-invariant term:

$$\mathcal{L}_\theta = 2\theta\, \tilde{F}^a_{\mu\nu} F^{\mu\nu,a}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

It is a total derivative

$$\mathcal{L}_\theta = 2\theta\, \partial_\mu\left(\varepsilon^{\mu\nu\alpha\beta} A^a_\nu F^a_{\alpha\beta}\right)$$

$\rightarrow$ does not contribute perturbatively but can contribute due to non-pert. effects.

θ = strong phase; generally $\theta \sim 1$

Turns out, $\theta \lesssim 10^{-11} \rightarrow$ strong CP problem

($F\tilde{F} \sim \vec{E}\cdot\vec{B}$ has $CP = -1$)

EDM until now unobservably small

Conserved currents of QCD $\mathcal{L}$

Eq. of M.

Gauge fields:

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^b_\mu F^{c\mu}{}_\nu = -g \bar{\psi} \gamma^\mu T^a \psi$$

Fermion field

$$(i\not{\partial} - m)\psi = -g \not{A}^a T^a \psi$$

$\mathcal{L}$ is gauge invariant $\Rightarrow$ global symmetry

$$\psi_i \to \psi_i + i\varepsilon^a T^a \psi \qquad \varepsilon^a \ll$$

$$A^a_\mu \to A^a_\mu - f^{abc} \varepsilon^b A^c_\mu$$

$\hookrightarrow N^2-1$ conserved currents for each α^a

$$J_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \qquad \phi_n = A^a_\mu, \psi$$

$$\hookrightarrow J^a_\mu = -\bar{\psi}\gamma^\mu T^a \psi + f^{abc} A^{\nu b} F^\mu{}_\nu{}^{,c}$$

with $\underline{\partial_\mu J^{\mu,a} = 0}$

However: the corresponding charges

$Q^a = \int d^3x \, J^{0,a}$ are gauge-dep.

(although $\partial_t Q^a = 0$)

→ No classical QCD current!

Matter current

$$j^a_\mu = -\bar{\psi}\gamma_\mu T^a \psi \quad \text{gauge-covariant}$$

but not conserved: $\partial^\mu j^a_\mu \neq 0$

Instead $D^\mu j^a_\mu = 0$

with $D_\mu j^a_\nu = \partial_\mu j^a_\nu + g f^{abc} A^b_\mu j^c_\nu$

→ Weinberg – Witten theorem

A theory w. a global non-abelian symmetry under which massless spin-1 particles are charged does not admit a gauge-invariant conserved current