Lecture 4

Massless spin-1 and gauge inv.

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + J_\mu A^\mu$$

$\hookrightarrow$ E.o.M. $(p^2 g^{\mu\nu} - p^\mu p^\nu) A_\nu = J^\mu$

Eq. not invertible : $\det(p^2 g^{\mu\nu} - p^\mu p^\nu) = 0$

Eigenvector p^μ, eigenv. $= 0$

The reason : gauge inv: $A^\mu \to A^\mu + \partial^\mu \alpha(x)$

Infinite nr. of vector fields obey EoM.

Gauge fixing: $\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \to$ into $\mathcal{L}$

Gauge invariance: ξ-independence of any correlation fn. $\langle \Omega | T\{O(x_1 \dots x_n)\} | \Omega \rangle$

$$= \frac{1}{Z[0]} \int \mathcal{D}A_\mu \, \mathcal{D}\varphi_i \, \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L}[A, \varphi_i]} \, O(x_1 \dots x_n)$$

for $O \to$ any G.I. collection of fields

$A^\mu \to A^\mu + \partial^\mu \alpha \longrightarrow \partial_\mu A^\mu \to \partial_\mu A^\mu + \Box \alpha$

or $\alpha = \frac{1}{\Box} \partial_\mu A^\mu$

Consider fn. $f(\xi) = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\partial^2\pi)^2}$

· transform $\pi \longrightarrow \pi - \alpha = \pi - \frac{1}{\Box}\partial_\mu A^\mu$

Shift $\longrightarrow$ $\int$ measure does not change

$$f(\xi) = \int D\pi \; e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2}$$

! Nothing changed, $f = f(\xi)$, not of A^μ

$\Downarrow$ $\times f(\xi)/f(\xi)$

$\langle\Omega| T\{O(x_1\ldots x_n)\}|\Omega\rangle$

$$= \frac{1}{Z[0]}\frac{1}{f(\xi)}\int D\pi \, DA_\mu \, D\varphi_i \, D\varphi_i^*$$

$$\times\; e^{i\int d^4x \left[\mathcal{L}[A,\varphi_i] - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2\right]}$$

$$\times\; O(x_1 \ldots x_n)$$

Stückelberg trick: $A^\mu = A'^\mu + \partial^\mu\pi$

Gauge transf. shift $\varphi_i \to e^{i\pi}\varphi_i'$

Again: it's a shift and $\int$ measure does not change

$\hookrightarrow$

$$\langle\Omega| T\{O(x_1\ldots x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\left[\frac{1}{f(\xi)}\int D\pi\right]$$

$$\times \int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L}[A,\varphi_i] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2} \, O(x_1 \ldots x_n)$$

Same prefactor for $Z[0]$ (no ext. fields)

$$Z[0] = \left[\frac{1}{f(\xi)} \int \mathcal{D}\kappa\right] \int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

$$\Downarrow$$

$$\langle \Omega | T\{O(x_1 \ldots x_n)\} | \Omega \rangle$$

$$= \frac{\int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L}} \, O(x_i)}{\int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L}}}$$

$$= \frac{\int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2} \, O(x_i)}{\int \mathcal{D}A^\mu \mathcal{D}\varphi_i \mathcal{D}\varphi_i^* \, e^{i\int d^4x \, \mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2}}$$

As announced: gauge fixing does not affect a correlation fn. of fields that is gauge-invariant

Gauge-invariance of the S-matrix was given in QFT I in perturbation theory

Consider Schwinger-Dyson Eq. in the context of path integrals

Scalar theory first

For $\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \mathcal{L}_{int}[\phi]$

$$(\Box_x + M^2)\langle\Omega|T\{\phi_x\phi_1 \dots \phi_n\}|\Omega\rangle$$

$$= \langle\Omega|T\{\mathcal{L}'_{int}[\varphi_x]\,\varphi_1 \dots \varphi_n\}|\Omega\rangle$$

$$- i\sum_j \delta^4(x-x_j)\langle\Omega|T\{\varphi_1 \dots \varphi_{j-1}\,\varphi_{j+1}\dots\varphi_n\}|\Omega\rangle$$

with

$$\mathcal{L}'_{int}[\varphi] = \frac{\partial}{\partial\varphi}\mathcal{L}_{int}[\varphi] \qquad \frac{g}{3!}\varphi^3 \to \mathcal{L}' = \frac{g}{2}\varphi^2$$

S.-D. Eq: m.e. of T-ordered products satisfy classical E.o.M. up to contact terms

1-point function

$$\langle\Omega|\hat{\phi}(x)|\Omega\rangle = \frac{-i}{Z[0]}\frac{\partial Z[J]}{\partial J(x)}\bigg|_{J=0}$$

$$= \frac{1}{Z[0]}\int D\varphi\, e^{i\int d^4y\left(-\frac{1}{2}\phi(\Box_y + M^2)\phi\right)}\phi(x)$$

Linear shift $\phi(x) \to \varphi(x) + \varepsilon(x)$

$$\langle 0|\hat{\phi}(x)|0\rangle = \frac{1}{Z[0]}\int D\phi\, e^{i\int d^4y\left[-\frac{1}{2}(\phi+\varepsilon)(\Box_y^2 + M^2)(\phi+\varepsilon)\right]}$$

$$\cdot(\phi+\varepsilon)$$

Expand to first order in ε

$$\langle 0|\hat{\phi}(x)|0\rangle = \frac{1}{Z[0]}\int D\phi\, e^{i\int d^4y\left(-\frac{1}{2}\phi(\Box_y + M^2)\phi\right.}$$

$$\times\left\{\phi(x)+\varepsilon(x)-i\,\phi(x)\int d^4z\,\tfrac{1}{2}\left[\phi(\Box_z+m^2)\varepsilon+\varepsilon(\Box_z+m^2)\phi\right]\right\}$$

by parts

$$\int d^4z\,\phi\,\Box_z\varepsilon=\int d^4z\,\varepsilon\,\Box_z\phi$$

$$\hookrightarrow\{\ldots\}=\phi+\varepsilon-i\,\phi\int d^4z\,\varepsilon(z)(\Box_z+m^2)\phi$$

The constant (in ε) term is already there, hence linear term should vanish for any ε

$$\downarrow\ \int d^4z\,\varepsilon(z)$$

$$\int D\phi\,e^{i\int d^4y\left(-\frac{1}{2}\phi(\Box_y+m^2)\phi\right)}\left[\phi(x)\Box_z\phi(z)+i\,\overset{\varepsilon(x)}{\hat{\delta}^4(x-z)}\right]=0$$

$$\Downarrow$$

$$(-i)^2(\Box_z+m^2)\left.\frac{\partial^2 Z[J]}{\partial J(x)\,\partial J(z)}\right|_{J=0}=-i\delta^4(x-y)\,Z[0]$$

or $(\Box_z+m^2)\langle\hat\phi(z)\hat\phi(x)\rangle=-i\delta^4(x-y)$

Green's fn. equation for Feynman prop.
<u>and</u> S-D eq. for 2-p. fn. in free scalar th.

Interacting theory

$$\mathcal{L}=-\tfrac{1}{2}\phi(\Box+m^2)\phi+\mathcal{L}_{int};\quad\Box\phi=\mathcal{L}_{int}'[\phi]$$

$\Downarrow$ all similar steps

$$(\Box_z+m^2)\langle\hat\phi(z)\hat\phi(x)\rangle=\langle\mathcal{L}_{int}'[\hat\phi(z)]\hat\phi(x)\rangle-i\delta^4(x-z)$$

Easy to generalize for more fields

$\hookrightarrow$ recover full S-D Eq.

Another way to write S-D Eq. is in the differential form

Start from generating functional

$$Z[J] = \int D\phi \, e^{i\int d^4z\left(-\frac{1}{2}\phi(\Box+m^2)\phi + \mathcal{L}_{int} + J\phi\right)}$$

Shift $\phi \to \phi + \varepsilon$, equate linear terms for any ε

$$\hookrightarrow (\Box_x + m^2) \int D\phi \, e^{i\int d^4y \, \mathcal{L}[\phi] + J\phi} \, \phi(x)$$

$$= \int D\phi \, e^{i\int d^4y \, \mathcal{L}[\phi] + J\phi} \, \mathcal{L}'[\phi(x)]$$

$$+ \int D\phi \, e^{i\int d^4y \, \mathcal{L}[\phi] + J\phi} \, J(x)$$

$\Downarrow$

$$\boxed{-i(\Box_x + m^2)\frac{\partial Z[J]}{\partial J(x)} = \left[\mathcal{L}'_{int}\left[\frac{-i\partial}{\partial J(x)}\right] + J(x)\right] Z[J]}$$

Similarly in QED

$$\mathcal{L} = \frac{1}{2} A_\mu \Box^{\mu\nu} A_\nu + \bar{\psi}(i\not{\partial}+m)\psi - eA_\mu \bar{\psi}\gamma^\mu\psi$$

w. $\Box^{\mu\nu} = \Box g^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu$

Classical EoM $\quad \Box_{\mu\nu} A^\nu = e j_\mu = e\bar{\psi}\gamma_\mu\psi$

Again, $A \to A + \varepsilon$

$\downarrow$

$$\Box^x_{\mu\nu} \langle A^\nu(x) A^\alpha(y) \bar{\psi}(z_1)\psi(z_2)\rangle$$

$$= e\langle j_\mu(x) A^\alpha(y) \bar{\psi}\psi\rangle - i\delta^4(x-y) g_\mu^\alpha \langle\bar{\psi}\psi\rangle$$

We will derive Ward-Takahashi ID

Consider correlation fn. $I_{12} = \langle \psi(x_1)\bar{\psi}(x_2)\rangle$

in a theory with a global symmetry under $\psi \to e^{i\alpha}\psi$

$$I_{12} = \int D\psi D\bar{\psi}\, e^{i\int d^4x [\bar{\psi}(i\not{\partial}+m)\psi + \dots]} \psi(x_1)\bar{\psi}(x_2)$$

any global-symm. int.

Field redefinition: local transformation

$$\psi(x) \to e^{-i\alpha(x)}\psi(x); \quad \bar{\psi} \to e^{i\alpha(x)}\bar{\psi}$$

The measure of path $\int$ is unchanged

$$i\bar{\psi}(x)\not{\partial}\psi(x) \longrightarrow i\bar{\psi}(x)\not{\partial}\psi(x) + \bar{\psi}\gamma^{\mu}\psi\,\partial_\mu\alpha(x)$$

$$\psi(x_1)\bar{\psi}(x_2) \to e^{-i\alpha(x_1)}e^{i\alpha(x_2)}\psi\bar{\psi}$$

The path integral can only change by a Jacobian (measure) since it is an $\int$ over <u>all</u> field configurations

$\Rightarrow$ to linear order in α

$$0 = \int D\psi D\bar{\psi}\, e^{iS}\left[i\int d^4x\,\bar{\psi}(x)\gamma^{\mu}\psi(x)\partial_\mu\alpha(x)\right] \cdot \psi(x_1)\bar{\psi}(x_2)$$

$$+\int D\psi D\bar{\psi}\, e^{iS}\left[-i\alpha(x_1)\psi(x_1)\bar{\psi}(x_2) + i\alpha(x_2)\psi(x_1)\bar{\psi}(x_2)\right]$$

$\Downarrow$ for any $\alpha(x)$

$$\int d^4x\,\alpha(x)\, i\partial_\mu\int D\psi D\bar{\psi}\, e^{iS}\,\bar{\psi}(x)\gamma^{\mu}\psi(x)\psi(x_1)\bar{\psi}(x_2)$$

$$= \int d^4x\,\alpha(x)\left[-i\delta^4(x-x_1) + i\delta^4(x-x_2)\right]\int D\psi D\bar{\psi}\, e^{iS}\psi_1\bar{\psi}_2$$

That is,

$$\boxed{\partial_\mu\langle j^{\mu}(x)\psi(x_1)\bar{\psi}(x_2)\rangle = -\delta^4(x-x_1)\langle\psi(x_1)\bar{\psi}(x_2)\rangle + \delta^4(x-x_2)\langle\psi(x_1)\bar{\psi}(x_2)\rangle}$$

$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \to$ the QED current

Schwinger–Dyson Eq. associated w.

charge conservation

! Non-perturbative relation between correlation functions

Fourier-transform the correlation fn.

$$\langle j^\mu(x)\, \psi(x_1)\, \bar\psi(x_2)\rangle$$

$$M^\mu(p,q_1,q_2) = \int d^4x\, d^4x_1\, d^4x_2\, e^{ipx} e^{iq_1x_1} e^{-iq_2x_2} \langle j^\mu(x)\, \psi(x_1)\, \bar\psi(x_2)\rangle$$

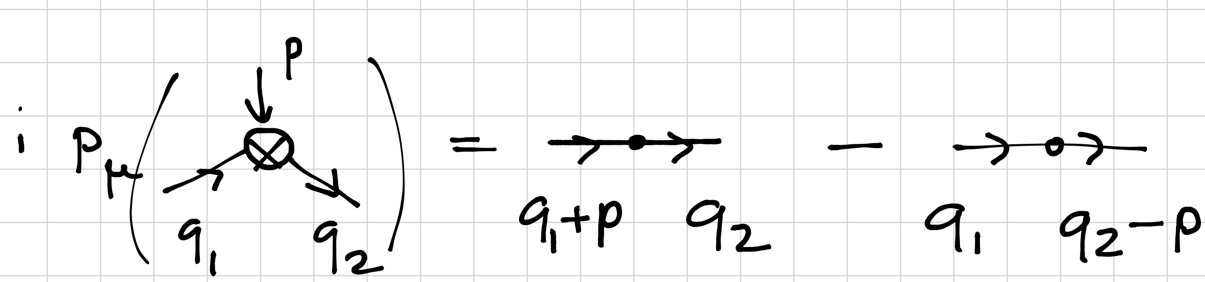
$$M_0(q_1,q_2) = \int d^4x_1\, d^4x_2\, e^{iq_1x_1} e^{-iq_2x_2} \langle \psi(x_1)\, \bar\psi(x_2)\rangle$$

$$M_0(p+q_1,q_2) = \int d^4x \int d^4x_1 \int d^4x_2\, e^{ipx} e^{iq_1x_1} e^{-iq_2x_2} \times \delta^4(x-x_1) \langle \psi(x_1)\, \bar\psi(x_2)\rangle$$

$$\Downarrow$$

$$\boxed{i p_\mu M^\mu(p,q_1,q_2) = M_0(q_1+p,q_2) - M_0(q_1,q_2-p)}$$

Charge conservation not based on perturbative expansion!



Momenta do not have to be on-shell
Momentum conservation is not implied

Can be generalized to any corr. fn.

$$M^{\mu\nu_1\dots\nu_b}(p, p_1, \dots p_b, q_1, \dots q_f)$$

$$= \int d^4x \dots e^{ipx} e^{ip_1x_1} e^{-iq_1y_1} \langle j^\mu(x)\, j^{\nu_1}(x_1) \dots \bar{\Psi}(y_1) \rangle$$

And $M_0^{\nu_1\dots\nu_b}(p_{1\dots b}, q_{1\dots f}) = \int d^4x_1 \dots e^{ip_1x_1} \dots \langle j^{\nu_1}(x_1) \dots \bar{\Psi}(y_1) \dots \rangle$

$\Downarrow$

$$ip_\mu M^{\mu\nu_1\dots\nu_b}(p, p_{1\dots b}, q_{1\dots f}) = \sum_{\text{incoming}} Q_i M^{\nu_1\dots b}(p_1 \dots q_i + p, \dots q_f)$$

$$- \sum_{\text{outgoing}} Q_i M^{\nu_1\dots b}(\dots q_i - p, \dots q_f)$$

$Q_i \rightarrow$ charge of the field $\Psi(y_i)$