Lecture 18

Anomalies – continuation

We have seen that sometimes symmetries of the Lagrangian are broken by quantum corrections $\longrightarrow$ anomalies

Example considered: massless QED

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\not{\partial} - e\not{A})\psi$$

The symmetry $U(1)_A \times U(1)_V$ has 2 Noeter currents

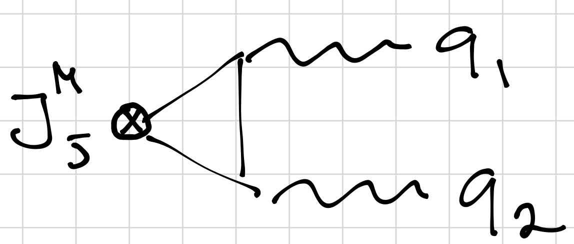
$$J^\mu = \bar{\psi}\gamma^\mu\psi \,, \quad J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$$

In presence of quantum corrections it is impossible for both currents to remain conserved:

$$\langle VVA \rangle = \langle \Omega | T\{ J_5^\mu J^\nu J^\alpha \} | \Omega \rangle$$

$$i\partial_\mu \langle J_5^\mu J^\nu J^\alpha \rangle = \frac{1}{4\pi^2} \varepsilon^{\nu\alpha\rho\sigma} q_{1\rho} q_{2\sigma}$$

$$\partial_\nu \langle J_5^\mu J^\nu J^\alpha \rangle = \partial_\alpha \langle J_5^\mu J^\nu J^\alpha \rangle = 0$$



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One of consequences: QED is not chiral: anomaly for vector current cancels exactly between ψ_L and ψ_R.

Path integral formulation:
anomaly arises when a symmetry of the action is not the symmetry of the integral measure

$$\langle O(x_1 \dots x_n) \rangle = \frac{1}{Z[0]} \int D\bar{\psi} D\psi$$

$$\exp\left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + i \bar{\psi} \not{D} \psi\right)\right] \cdot O$$

with O any gauge-invariant op.

Under $\psi \to e^{i\alpha(x)} \psi$, action

$$\bar{\psi} \not{\partial} \psi \to \bar{\psi} \not{\partial} \psi + i \bar{\psi} \gamma^\mu \psi \, \partial_\mu \alpha$$

$\int$ measure $\quad D\bar{\psi} D\psi \to |J|^{-2} D\bar{\psi} D\psi$

(inverse Jacobian for fermions)

$$J = \exp\left(i \int d^4x \, \alpha(x)\right) \longrightarrow |J|^2 = 1$$

and the path integral remains unchanged

The difference:

$$0 = \frac{1}{Z[0]} \int d^4z \, \alpha(z) \int D\bar{\psi} D\psi \exp\left[i S[A, \psi]\right]$$

$$\cdot \frac{\partial}{\partial z^\mu} \left[\bar{\psi}(z) \gamma^\mu \psi(z)\right] \, O$$

(integration by parts)

$\hookrightarrow \partial_\mu \langle J^\mu(x)\, O(x_1 \ldots x_n)\rangle = 0$

Vector current conserved.

For axial transformation $\psi \mapsto e^{i\beta(x)\gamma_5}\psi$

the measure is not invariant!

$$J = \exp\left[i \int d^4x\, \beta(x)\, \mathrm{Tr}\, \gamma_5 \right] = 0$$

To regulate the divergence introduc a <u>gauge-invariant</u> regulator

$$\exp\left(\frac{(i\not{D})^2}{\Lambda^2}\right)$$

and take limit $\Lambda \to \infty$ at the end

$$\hookrightarrow J = \exp\left[i \int d^4x\, \beta(x)\, \mathrm{Tr}\left(\gamma_5 e^{-\frac{\not{D}^2}{\Lambda^2}}\right)\right]_{\Lambda\to\infty}$$

$$\not{D}^2 = D^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}$$

The first non-vanishing contribution to trace comes at order $\frac{1}{\Lambda^4}$:

$$\mathrm{Tr}\, \gamma_5 \sigma^{\mu\nu} = 0$$

$$\mathrm{Tr}\, \gamma_5 \sigma^{\mu\nu}\sigma^{\alpha\beta} = 4i\varepsilon^{\mu\nu\alpha\beta}$$

$$\longrightarrow \beta\, \mathrm{Tr}(\quad)_{\Lambda\to\infty} = \frac{e^2}{2}\beta(x)\,\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$\bullet \lim_{\Lambda\to\infty}\left[\frac{1}{\Lambda^4} e^{\left(-\partial^2/\Lambda^2 + O(e^2)\right)}\right]$$

Some subtleties : sandwich between states $|x\rangle$, $|k\rangle$ of 1P Hilbert space

$$\hookrightarrow \quad \frac{1}{\Lambda^4}\langle x| e^{-\partial^2/\Lambda^2}|x\rangle = \frac{1}{\Lambda^4}\int \frac{d^4k}{(2\pi)^4} e^{k^2/\Lambda^2}$$
$$= \frac{i}{16\pi^2}$$

$$\Downarrow$$

$$\int D\bar{\Psi} D\Psi DA \exp[iS]\, O(\ldots)$$

$$\longrightarrow \int D\bar{\Psi} D\Psi DA \exp\Big[i\int d^4x \Big(\mathcal{L} - J^5_\mu \partial^\mu \beta + \beta \frac{e^2}{16\pi^2}\tilde{F}F\Big]\, O(\ldots)$$

$$\Downarrow$$

For arbitrary $\beta(x)$ and O

$$\boxed{\partial_\mu J^\mu_5 = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}}$$

In this form it is explicit that the anomaly is exact and obtains no perturbative corrections

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That the symmetry of the Lagrangian is destroyed by quantum corrections should be disturbing if the symmetry is gauged.

In fact, for a gauge field theory to be consistent, gauge anomalies should cancel (unlike global anomalies)

Let's observe how this cancellation works in the Standard Model.

$$SU(3)_{QCD} \times SU(2)_{weak} \times U(1)_Y$$

Respective currents J^μ_k, $k = s, w, Y$ should be non-anomalous

$$\longrightarrow \quad \partial_\mu \langle J^\mu_a J^\nu_b J^\alpha_c \rangle = 0$$

1. $a = b = c = Y \qquad (U(1)^3_Y)$

$$\partial_\mu J^\mu_Y = \Big(\sum_{\text{left}} Y_\ell^3 - \sum_{\text{right}} Y_r^3 \Big) \frac{g'^2}{32\pi^2} \tilde{B}^{\mu\nu} B_{\mu\nu}$$

As for QED, $Q_L - Q_R$ always appears in vector anomaly

$$\Rightarrow \boxed{\big(\underset{e_L+\nu_L}{2 Y_L^3} - Y_e^3 - Y_\nu^3\big) + \underset{N_C}{3}\big(\underset{u_L+d_L}{2 Y_Q^3} - Y_u^3 - Y_d^3\big) = 0}$$

SM hypercharges:

$$Y_L = -\frac{1}{2}, \quad Y_e = -1, \quad Y_\nu = 0$$

$$Y_Q = \frac{1}{6}, \quad Y_u = \frac{2}{3}, \quad Y_d = -\frac{1}{3}$$

$$2 Y_L^3 - Y_e^3 = \frac{3}{4}$$

$$(2 Y_Q^3 - Y_u^3 - Y_d^3) = \frac{1}{108} - \frac{8}{27} + \frac{1}{27} = -\frac{1}{4}$$

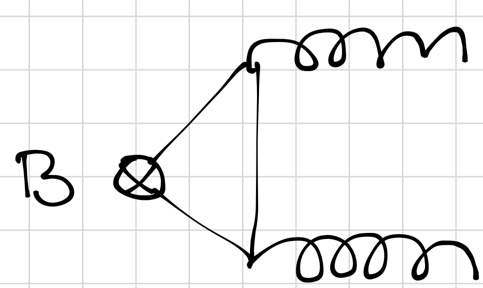
$$\hookrightarrow \quad \frac{3}{4} + 3\left(-\frac{1}{4}\right) = 0 \qquad \text{OK!}$$

In case of non-abelian symmetries need to account for commutation relations of group generators

$$J^{\mu,a} = \bar{\psi} T^a \gamma^\mu \psi$$

Because $\mathrm{tr}\, T^a = 0$ anomaly $\sim SU(N)\, U(1)^2 = 0$

$SU(3)^2\, U(1)_Y$



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$$\mathrm{tr}\, T^a T^b = \frac{1}{2} \delta^{ab}$$

$$\hookrightarrow \quad \boxed{2 Y_Q - Y_u - Y_d = 0}$$

$$2 \cdot \frac{1}{6} - \frac{2}{3} + \frac{1}{3} = 0 \qquad \text{OK}$$

$SU(2)^2\, U(1)_Y \longrightarrow$ involves only L-fermions

$\Downarrow$

$$\boxed{Y_L + 3Y_Q = 0}$$

$-\frac{1}{2} + 3 \cdot \frac{1}{6} = 0$ OK

$SU(2)^3_w \sim tr\left[\tau^a\tau^b\tau^c + \tau^a\tau^c\tau^b\right] = 0$

$\hookrightarrow$ no such anomaly

$SU(3)^3_s = 0$ since gluons couple to q_L and q_R equally (like QED)

A more general case for hypercharges (an alternative to SM)

$2Y_L = x + 2y$, $Y_e = x + y$, $Y_\nu = y$

$2Y_Q = -\frac{1}{3}(x+2y)$, $Y_u = -\frac{1}{2}(2x+y)$, $Y_d = \frac{1}{2}(x-y)$

SM: $y = 0$, $x = -1$

Importantly, $Y_L + 3Y_Q = 0$ always
proton and electron charges are exactly equal and opposite

What if yet another $U(1)_{Y'}$ exists that couples to SM fermions?

$\rightarrow$ Anomalies $U(1)^3_{Y'}$, $U(1)^2_{Y'} U(1)_Y$ and $U(1)_{Y'} U(1)^2_Y$ should cancel

$\Rightarrow$ the same equations as for Y_i but a different solution (x, y)

$x = 0, \quad y = -1$

$Y'_L = Y_e = Y_\nu = -1 \longrightarrow$ lepton number L

$Y'_Q = Y'_u = Y'_d = +\frac{1}{3} \longrightarrow$ baryon number B

$\hookrightarrow$ this group is $U(1)_{B-L}$

This symmetry is not gauged — global

The currents corresponding to B and L are anomalous!

Only nonzero: $SU(2)^2 U(1)_B$; $SU(2)^2 U(1)_L$

$$\partial_\mu J^\mu_B = n_g^q \frac{g^2}{32\pi^2} \tilde{W}^{\mu\nu}_a \cdot W^a_{\mu\nu}$$

(each quark has $B = +\frac{1}{3} \rightarrow$ compute for a baryon $|qqq\rangle$ with $B = 1$)

For leptons $\rightarrow$ use $L = 1$

$$\Rightarrow \partial_\mu J^\mu_L = n_g^\ell \frac{g^2}{32\pi^2} \tilde{W}^{\mu\nu}_a W^a_{\mu\nu}$$

! Number of generations for leptons and quarks has to be the same

$$\boxed{\partial_\mu J^\mu_{B-L} = 0}$$

B and L cannot be associated to any gauge symmetry assosiated w. B or L But possible to have gauge boson associated to B-L

This is a situation of grand unification (leptons acquire a color and are treated together with quarks)

$$SM = SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$$

GUT (grand unification theory)

Then, this opens a channel for proton decay $p \to \pi^0 e^+$ or hydrogen atom annihilation

Proton is stable ($\tau_p > 10^{33}\,yr$)

$\hookrightarrow M_{B-L} \gtrsim 10^{16}\,GeV$

The very possibility of violating B and L conservation (holds classically) but keeping B-L intact offers a solution to Matter-Antimatter asymmetry observed in the Universe: ordinary matter is dominated by p, e^- and n, whereas the fraction of $\bar{p}, \bar{n}, e^+$ is tiny.

Then, for our Universe to be created out of symmetric Big Bang, B and L should be violated, but in exactly correlated way, $\Delta B - \Delta L = 0$

Sakharov's criteria for baryogenesis:

1. B-violation
2. C and CP-violation
3. Universe should be out of thermal equilibrium

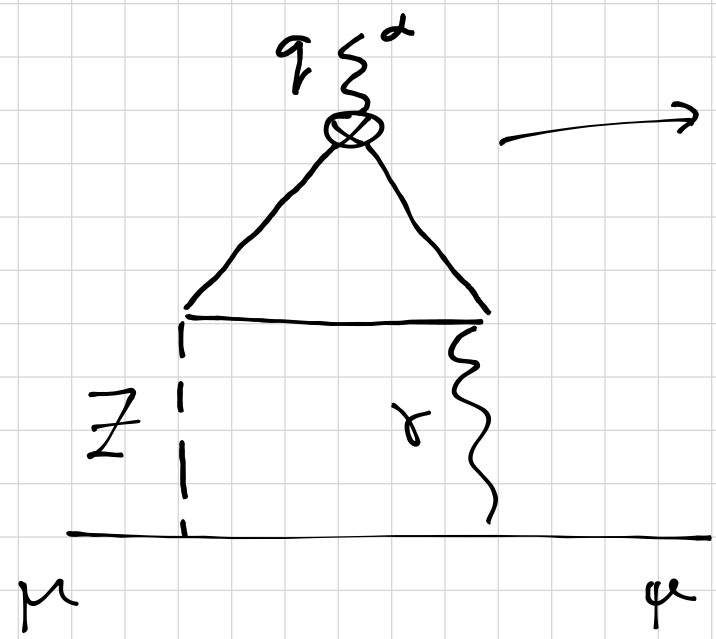
All these ingredients are present in the Standard model, but are not sufficient to explain the Universe that we observe.

To sum up: anomalies are a very

important aspect of gauge field theory. It took bright minds to derive and understand them.

While most work was done in 1960–1970, there are examples when sound physicists came to wrong conclusions by ignoring anomaly cancellation.

E.g. 2-loop EW contribution to muon $g-2$:



$$\rightarrow \bar{\mu} \, \frac{i\sigma^{\alpha\beta} F_{\alpha\beta}}{4 m_\mu} \, \mu$$

$+$ crossed

This graph is UV-sensitive, and in each fermion loop $\log(M_Z^2/m_f^2)$ is generated; a tumult was created when a large "missed" contribution was found

→ subsequently shown that the anomaly cancellation in SM forces

the coeff in front of $\ln M_Z^2$
to be exactly $\emptyset$ upon summing
over leptons and quarks.
The only sizable contribution is
then generated by the flavor sym.
breaking by the fermion masses.