Lecture 16

Fermion masses

In pure QED $\mathcal{L}_m = -m\bar{e}e = -m(\bar{e}_R e_L + \bar{e}_L e_R)$

Dirac mass term

$\rightarrow$ expelicitely breaks SU(2) symmetry

Can write it in a SU(2)-invariant way,

$$\mathcal{L}_{Yukawa} = -y \bar{L} H e_R + h.c.$$

upon SSB $\Rightarrow -y \frac{v}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L)$

with $y = \frac{\sqrt{2} m}{v}$

This way we only give masses to the lower members of doublets

Define $\tilde{H} = i\sigma_2 H^* = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$

H has hyperchange $\frac{1}{2}$

$\tilde{H}$ $-\frac{1}{2}$

$\Downarrow$

Generically, $-y^d \bar{L} H d_R - y^u \bar{L} \tilde{H} u_R$

! $\bar{L}H$ and $\bar{L}\tilde{H}$ are SU(2) invariant

When extending the

Yukawas to 3 generations, the Yukawa Matrix is not necessarily diagonal!

$$\mathcal{L}_{mass} = - Y^{d}_{ij} \bar{Q}^{i} H d^{j}_{R} - Y^{u}_{ij} \bar{Q}^{i} \tilde{H} u^{j}_{R} + h.c.$$

There always exist unitary matrices U_u, U_d and diagonal matrices M_u, M_d such that

$Y_d Y_d^{+} = U_d M_d^2 U_d^{+}$ and $Y_u Y_u^{+} = U_u M_u^2 U_u^{+}$.

On the other hand, we can write $(KK^{+} = 1)$

$Y_d = U_d M_d K_d^{+}$, $Y_u = U_u M_u K_U^{+}$

After symmetry breaking $(H \rightarrow v/\sqrt{2})$

$$\mathcal{L}_{mass} = -\frac{v}{\sqrt{2}} \left[\bar{d}_L U_d M_d K_d^{+} d_R + \bar{u}_L U_u M_u K_u^{+} u_R \right] + h.c.$$

and absorb $U_{d,u}$, $K_{d,u}$ into the quark states. definition $\longrightarrow$ go to mass basis where the mass terms are diagonal

$$\mathcal{L}_{mass} = - m^{d}_{j} \bar{d}^{j}_{L} d^{j}_{R} - m^{u}_{j} \bar{u}^{j}_{L} u^{j}_{R} + h.c.$$

with $m^{u,d} = \frac{v}{\sqrt{2}} M_{u,d}$

The original quark states were in flavor basis, where the SU(2) boson W^a couplings were flavor-diagonal. The diagonal terms in the kinetic term connect $\bar{u}_L$ with u_L, $\bar{d}_L$ with d_L and $U_d^+ U_d = U_u^+ U_u = \mathbb{1}$ drops.

However, off-diagonal terms $\sim W^\pm$ connect $\bar{u}_L$ and d_L etc, and $U_u^+ U_d \neq 1$

$\Downarrow$

$$\mathcal{L}_{mass-basis} = \frac{e}{\sin\theta_W} Z_\mu J_Z^\mu + e A^\mu J_\mu^{EM}$$
$$- m_j^d (\bar{d}_L^j d_R^j + \bar{d}_R^j d_L^j) - m_j^u (\bar{u}_L^j u_R^j + \bar{u}_R^j u_R^j)$$
$$+ \frac{e}{\sqrt{2}\sin\theta_W} \left[W_\mu^+ \bar{u}_L^i \gamma^\mu V^{ij} d_L^j + W_\mu^- \bar{d}_L^i \gamma^\mu (V^+)^{ij} u_L^j \right]$$

where

$$V = U_u^+ U_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \qquad VV^+ = V^+ V = \mathbb{1}$$

Cabibbo – Kobayashi – Maskawa (CKM) matrix

V is generally complex;

Real V is an $O(3)$ rotation matrix with 3 Euler's angles

The Lagrangian has a global $U(1)^6$ symmetry

$$d_{L,R}^j \to e^{i\alpha_j} d_{L,R}^j \quad ; \quad u_{L,R}^j \to e^{i\beta_j} u_{L,R}^j$$

We can always rotate some phases to $\emptyset$

Since if $\alpha_j = \beta_j$ V remains unchanged, one phase cannot be eliminated

$\Rightarrow$ in total, 3 independent angles
+ 1 phase

This remaining phase is specific for 3 generation case : 2 generation admits no phase!

Can define 3 angles rotating in the plane $ij = 12, 13, 23$ $\quad \theta_{12}, \theta_{13}, \theta_{23}$ and one phase δ.

$\Downarrow$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Originally, Cabibbo introduced only one angle, $\theta_{12} \equiv \theta_C$ (Cabibbo angle)
to reconcile the strength of weak interaction in lepton and quark sector.

$$\frac{1}{\tau_{\mu e}} = G_\mu^2 \times \text{masses} \times \text{phase space}$$

$$\frac{1}{\tau_{n,\pi}} = G_{ud}^2 \times \text{masses} \times \text{phase space}$$

$$\frac{1}{\tau_{K,\Sigma}} = G_{us}^2 \times \text{masses} \times \text{phase space}$$

β decay rates of light-flavor hadrons were found very similar to muon but too low (including RC $\longrightarrow \sim 5\%$ deficit)
weak decays of strange hadrons were even slower (hence the name "strange")
Cabibbo assumed that a single-angle mixing among two flavors

$$\hookrightarrow \quad G_\mu \longrightarrow \begin{pmatrix} G_{ud} & G_{us} \\ -G_{us} & G_{cs} \end{pmatrix} = G_\mu \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix}$$

restoring the quark-lepton universality.

CP violation observed in the decay of K^0 and $\overline{K}^0$ indicated that CKM matrix had a phase $\Longrightarrow$ 3rd generation was predicted before it was discovered.
The complete CKM matrix was developed by Kobayashi and Maskawa.

Experimentally: $s_{13} \ll s_{23} \ll s_{12} \ll 1$

$\longrightarrow$ Wolfenstein parametrization:

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}$$

$$s_{23} = A\lambda^2 = \lambda \left| \frac{V_{cb}}{V_{us}} \right|$$

$$s_{13} e^{i\delta} = V_{ub}^* = A\lambda^3(\rho + i\eta)$$

Expanding to order λ^3,

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

Unitarity of CKM matrix — consequence of the universality of weak interaction and completeness of the standard model

$$\sum_i V_{ij} V_{ik}^* = \delta_{jk} \qquad \sum_j V_{ij} V_{kj}^* = \delta_{ik}$$

Prominent example: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

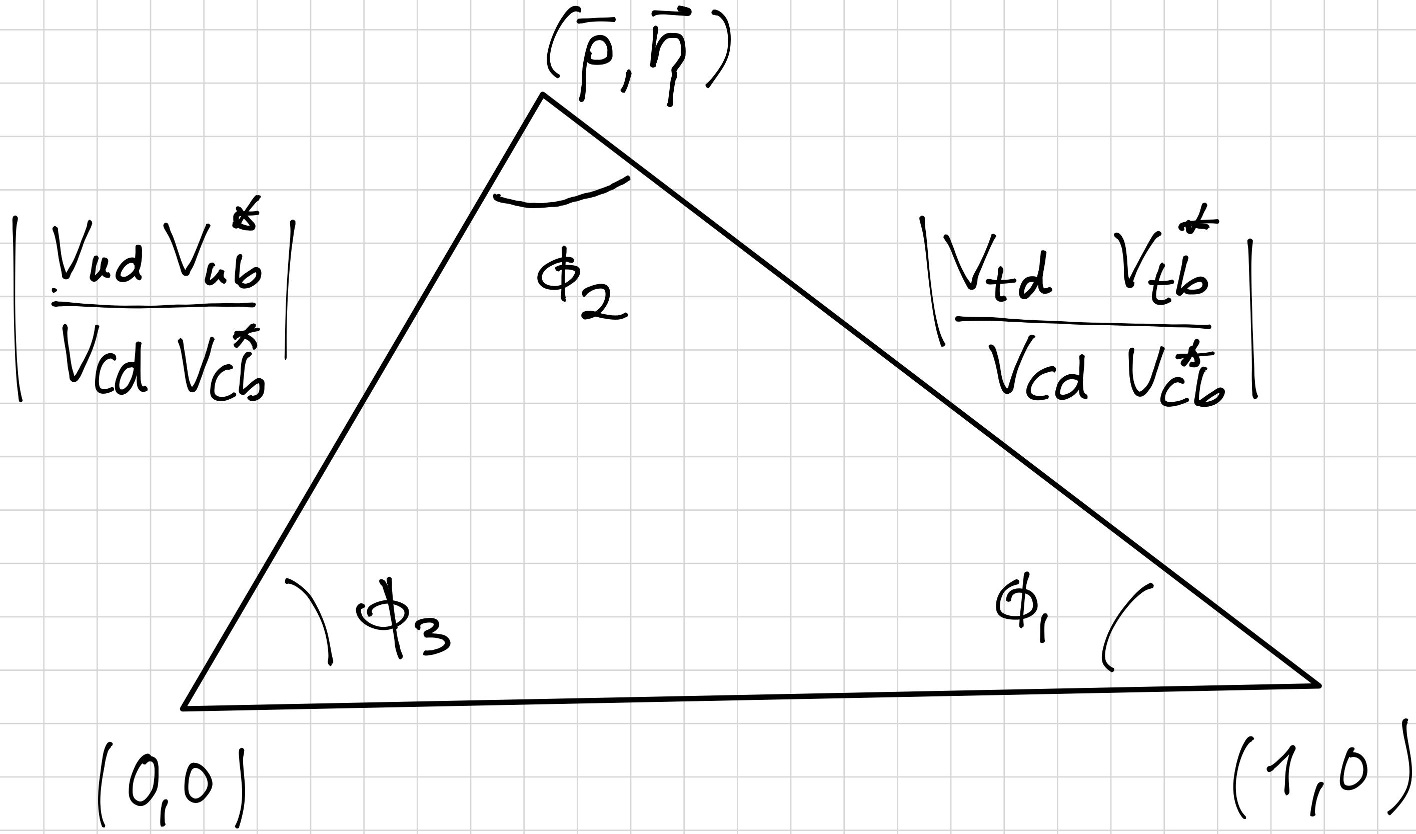
PDG 2021
$$|V_{ud}| = 0.97373(11)_{\text{Exp.}}(9)_{RC}(27)_{\text{Nuclear}}$$
$$|V_{us}| = 0.2243(8)_{\text{Exp. + lattice}}$$
$$|V_{ub}| \cong 4 \cdot 10^{-3}$$

Currently observed 2σ deficit:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985(6)_{V_{ud}}(4)_{V_{us}}$$

Unitarity triangle

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$$



$$\phi_1 + \phi_2 + \phi_3 = \left(179^{+7}_{-6}\right)^\circ$$

$\sin\theta_{12} = 0.22650(48)$

$\sin\theta_{23} = 0.04053^{(83)}_{(61)}$

$\sin\theta_{13} = 0.00361^{(11)}_{(9)}$

$\delta = 1.196^{(45)}_{(43)}$

From global fit

To summarize: deviation from unitarity would signal a problem with Standard Model (missing particles, missing interactions)

Unitarity triangle requires combining huge data set from nuclear β decays (V_{ud} – light quarks) to collider physics.

Extremely complicated – in fact, difficult to achieve high precision

Best known unitarity constraint: top-row or Cabibbo unitarity $|V_{ud}|^2 + |V_{us}|^2 = 1$

$$\cos^2\theta_C + \sin^2\theta_C = 1$$

Relies on "just" $0^+ \to 0^+$ nuclear decays (15 best measured)

and Kaon decays (~6 channels)

A very active field of research
Local group contributed to establishing the unitarity deficit
Whenever a deviation is found model builders look for a mechanism of New Physics that can explain this observation without destroying the rest.

For quarks we know quark masses better than their weak interaction (strong interaction is stronger)
Then, it was convenient to work in the mass basis → CKM matrix

For neutrinos it's the opposite:
we only observe them via weak interaction and want to determine masses.

$$\mathcal{L}_{mass} = -Y^e_{ij}\bar{L}^i H e^j_R - Y^\nu_{ij}\bar{L}^i \tilde{H}\nu^j_R - iM_{ij}(\nu^i_R)^c \nu^j_R + h.c.$$

Second term: Dirac mass term (L−R)
Third term: Majorana mass term (lowest dim.)
Charge-conjugate spinor

$$\nu^c = \nu^T \sigma_2$$

Interestingly, ν_R is not charged under any interaction → has no EW quantum nr.

The Majorana mass term violates lepton number conservation (it is certainly a problem in SM, but we don't know if it is in general)

(One may have a Majorana mass term without ν_R, but at dim. 5:

$$\mathcal{L}_{m5} = -\tilde{M}_{ij}(\bar{L}^i\tilde{H})(\tilde{H}L^j)^+$$
)

Back to dim-3 Majorana MT:
If introducing notation (recall $\Psi = \begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}$)

$$\Psi_L = \begin{pmatrix}\nu_L\\ i\sigma_2\nu_L^*\end{pmatrix} \quad \text{and} \quad \Psi_R = \begin{pmatrix} i\sigma_2\nu_R^*\\ \nu_R\end{pmatrix}$$

$$\mathcal{L}_{\nu\text{-mass}} = -m\,\bar{\Psi}_L \Psi_R - \frac{M}{2}\,\bar{\Psi}_R \Psi_R$$

Now, Ψ_L and Ψ_R have both upper and lower component and can mix!

Mass eigenstates diagonalize the mass matrix $\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$

These physical masses are

$$\sqrt{m^2 + \frac{1}{4}M^2} \pm \frac{1}{2}M \nearrow M \gg \quad \searrow \frac{m^2}{M} \ll$$

This is called see-saw mechanism that explains why ν-masses are so small

If $M \approx M_{Plank} \cong 10^{19}$ GeV

Then for Dirac masses $m \sim 100$ GeV (EW scale) $m_{eight} \sim \frac{m^2}{M} \sim 10^{-6}$ eV

Why Plank scale? — because no particles above EW scale have been observed

If light ν masses are large it may indicate that there is some New Physics in between.

In general: if Majorana mass term

exists it will signal existence of a mechanism of lepton number violation (beyond SM) BSM

In any case, ν masses are already a solid indication that BSM physics exists!

Why bothering much about ν masses? They could be just zero... but they're not

As for quarks, mixing is due to off-diagonal $W^\pm$ couplings

$$\mathcal{L}_{\nu W} = -\frac{g}{\sqrt{2}}\left(\bar{e}_L \not{W} \nu_{Le} + \bar{\mu}_L \not{W} \nu_{L\mu} + \bar{\tau}_L \not{W} \nu_{L\tau}\right) + h.c.$$

Mass eigenstates are related to flavor ones via

$$\begin{pmatrix} \nu_{Le} \\ \nu_{L\mu} \\ \nu_{L\tau} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & \\ & e^{i\frac{\alpha_{12}}{2}} & \\ & & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \begin{matrix} \rightarrow m_1 \\ \rightarrow m_2 \\ \rightarrow m_3 \end{matrix}$$

$\delta \rightarrow$ Dirac phase

Other two phases appear in Majorana case only.

Neutrinos are observed via lepton appearance (ν-disappearance) $\quad \nu_e + n \rightarrow p + e^-$

$\nu_\mu + p \rightarrow n + \mu^+$ etc.

One measures the flavor ν states.

Then $\rightarrow$ model the initial flux of ν's

model the conversion probability

count the number of leptons

$\longrightarrow$ compare

Solar neutrino problem:

(thermo) nuclear processes in the Sun

$\longrightarrow$ a flux of neutrinos of e-type can be predicted in the "Standard Solar Model"

$\longrightarrow$ the number of e^- appeared in detectors on Earth was $\sim \frac{1}{3}$ of what was expected

Solution: neutrino oscillations

Plane-wave propagation

$$|\nu_j(t)\rangle = e^{i\vec{p}_j\vec{x} - iE_j t}\,|\nu_j(0)\rangle$$

ν's are ultrarelativistic $\longrightarrow E_j \simeq p_j + \frac{m_j^2}{2p_j}$

Then, at a distance L,

$$|\nu_j(L)\rangle \cong e^{-i\frac{m_j^2 L}{2E}} |\nu_j(0)\rangle$$

Different mass states gain a different phase by the time they arrive at L.

Through mixing matrix U mass states mix into flavor states, leading to oscillations:

Conversion probability flavor $\alpha \to$ flavor β at distance L is

$$P_{\alpha\to\beta} = |\langle \nu_\beta | \nu_\alpha(L)\rangle|^2 = \left|\sum_j U_{\alpha j}^* U_{\beta j} e^{-i\frac{m_j^2 L}{2E}}\right|^2$$

For 2-flavor oscillations (1 angle θ)

$$P_{\alpha\to\beta,\ \beta\neq\alpha} = \sin^2(2\theta)\ \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$\Delta m^2 = m_2^2 - m_1^2$$

For 3-flavor

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{j>k} \mathrm{Re}\{U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k}\} \sin^2\left(\frac{\Delta_{jk} m^2 L}{4E}\right)$$

$$+ 2\sum_{j>k} \mathrm{Im}\{U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k}\} \sin\left(\frac{\Delta_{jk} m^2 L}{4E}\right)$$

Second term CP-violating: it changes sign for $\nu_\alpha \to \nu_\beta$ and $\bar\nu_\alpha \to \bar\nu_\beta$ conversion.