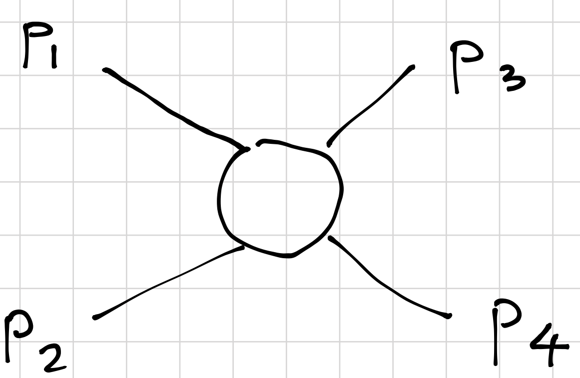
Lecture 12

Theory of complex angular momentum,
Regge theory
Veneziano model
Strings

1. Consider elastic scattering of 2 identical particles of mass m, spin 0 (for simplicity)



$$s = (p_1 + p_2)^2$$
$$u = (p_2 - p_3)^2 \qquad s + u + t = 4m^2$$
$$t = (p_1 - p_3)^2$$

c. m. kinematics: (s - channel)

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \longrightarrow E = \frac{\sqrt{s}}{2}$$
$$|p| = \frac{\sqrt{s - 4m^2}}{2}$$
$$t = -2|p|^2(1 - \cos\theta)$$
$$\cos\theta_s = 1 + \frac{t}{2|p|^2} = 1 + \frac{2t}{s - 4m^2}$$

Scattering described by a single ampl.
$A(s,t)$

It permits a standard partial wave expansion

$$A(s,t) = \sum_{n=0}^{\infty} (2n+1)\, g_n(s)\, P_n(\cos\theta_s)$$

Similarly, in the t-channel

$$A(s,t) = \sum_{n=0}^{\infty} (2n+1)\, f_n(t)\, P_n(\cos\theta_t)$$

$$\cos\theta_t = 1 + \frac{2s}{t-4m^2} \equiv z \qquad |z| \leq 1$$

What is the asymptotics of A at $s \to \infty$

This corresponds to asking the limit of A at $z \to \infty$ $\to$ unphysical region

We need to analytically continue the PW expansion to unphysical values of z.

From the general properties of PW amplitudes f_n: while $P_n(z) \sim z^n$, for $n \gg$ f_n drop very fast

Then, we should expect that only PW with $n \leq n_0(t)$ [we allow n_0 to depend on t] contribute much to $z \to \infty$ asymptotics.

$$A(s,t) = \sum_{n=0}^{n_0(t)} (2n+1)\, f_n(t)\, P_n(z) + \ldots$$

Then, $A(s,t)_{s\to\infty} \sim z^{n_0(t)} \sim s^{n_0(t)}$

How can we determine $n_0(t)$?

Here we turned around the question!

Rather than asking, what the structure of the amplitude in s and t, we explore its structure in the <u>complex angular</u>

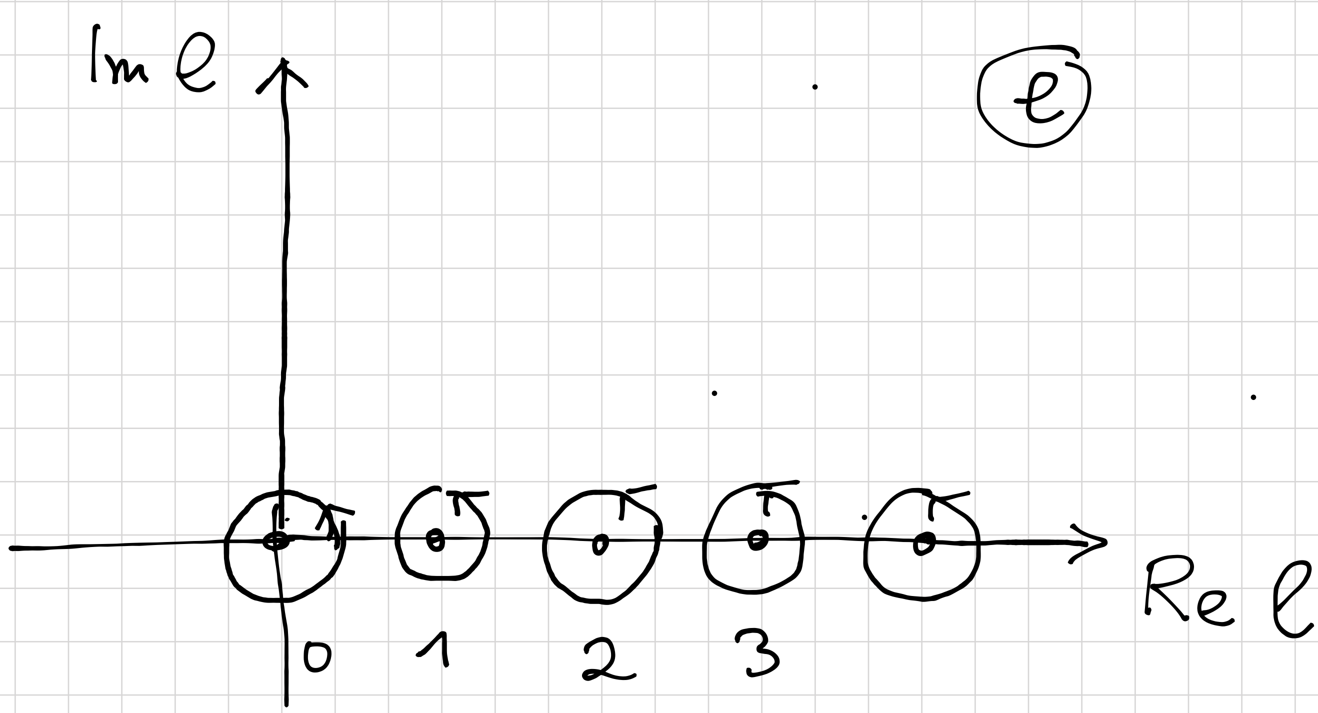
momentum ℓ!

Analytical continuation $f_n(t) \longrightarrow f_\ell(t)$

$$f_\ell(t) = f_n(t) \quad \text{for } \ell = n = 0, 1, 2 \ldots$$

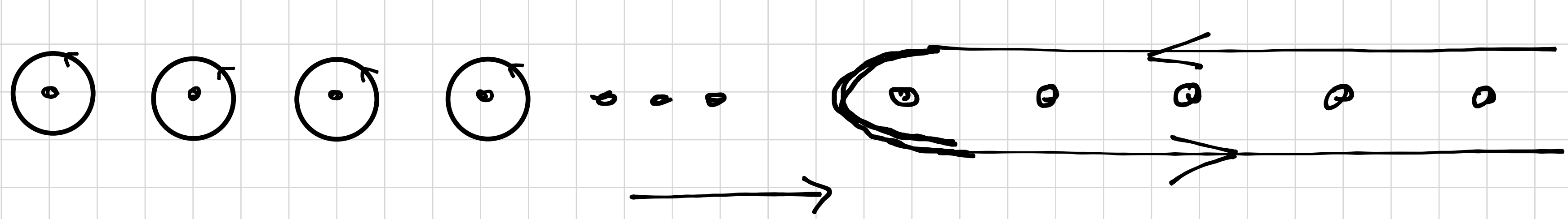
Sommerfeld – Watson representation

$$A(s,t) = \frac{1}{2i} \int_C \frac{d\ell}{\sin(\pi \ell)} f_\ell(t) P_\ell(-z)$$



Legendre polynomials are analytic in ℓ
The original PW expansion recovered by summing over residues at simple poles $\frac{1}{\sin \pi \ell}$ for integer $\ell = n$, and with $P_n(-z) = (-1)^n P_n(z)$

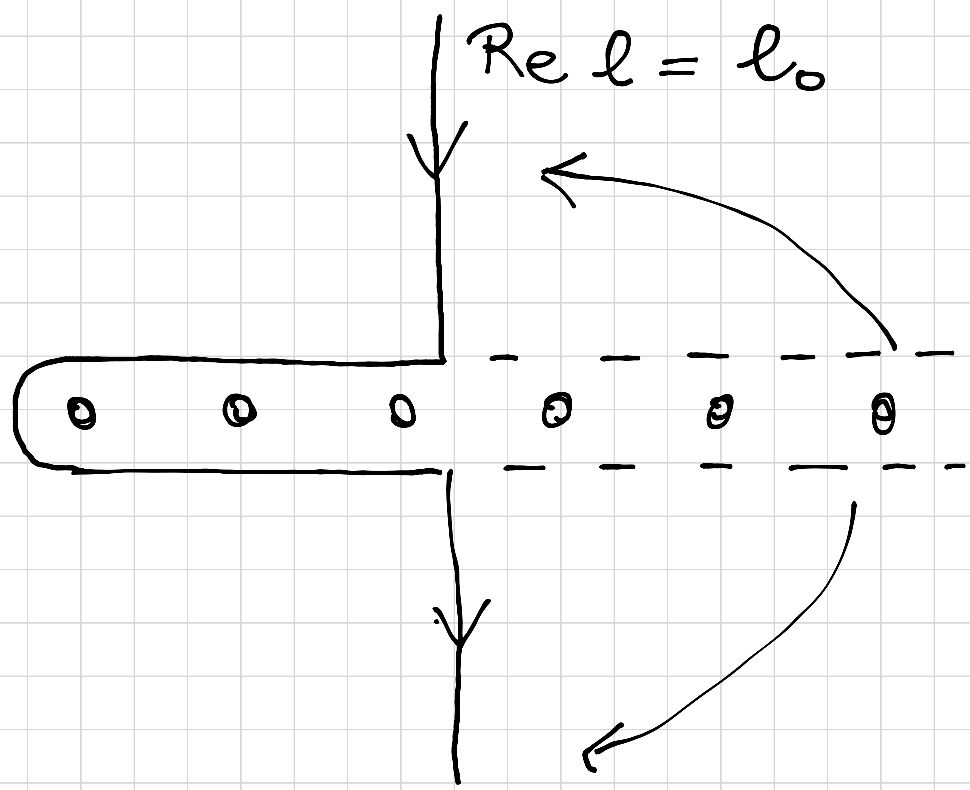
Now deform the contour as



assuming that there are no singularities in the vicinity of $\ell = n$

$\Longleftrightarrow$ assuming that PW amplitudes f_ℓ have no singularities between n and $n+1$; the amount by which we go into complex values of ℓ can be $\sim i\varepsilon$

Then we wish to further deform the contour:



$\mathrm{Re}\,\ell \leq \ell_0 = $ const.

$$P_\ell(z) \sim z^\ell = z^{\mathrm{Re}\,\ell + i\,\mathrm{Im}\,\ell}$$

$$A(s,t) \sim |P_\ell(z)| \lesssim z^{\ell_0}, \quad z \to \infty$$

Clearly, the bound becomes stronger, the farther to the left we go. What can make us stop? When can we go no further to the left?

! If $f_\ell(t)$ is an analytical function of ℓ $\rightarrow$ it has to have singularities somewhere on the complex plane ℓ !

Then ℓ_0 is the position of the rightmost singularity of $f_\ell(t)$, $\ell = \alpha(t)$

Before we get to this singularity: what are prerequisites for the manipulations with the contours?
E.g., when are we allowed to close the contour to $\pm i\infty$?

Carlson's theorem: $|f_\ell| < e^{|\ell|\pi}$, $|\ell| \to \infty$
$\rightarrow$ sufficient condition for extrapolating a function from its value at integer points on the real axis to the entire complex plane.
This follows from observing that

$$\left|\frac{1}{\sin \pi\ell}\right|\Bigg|_{\ell \to \pm i\infty} = \left|\frac{2i}{e^{\mp\pi|\ell|} - e^{\pm\pi|\ell|}}\right| \sim e^{-\pi|\ell|}$$

A formal proof is possible (I skip that ☺)

So now the question boils down to: what are the singularities of the

PW amplitudes as fn. of the complex angular momentum ℓ?

Generally: analytical functions have pole and cuts

Regge theory (after Emilio Regge who studied the analytical structure of solutions of radial Schroedinger eq.)

deals with the high-energy behavior of scattering amplitudes as a consequence of the positions of its singularities in the complex angular momentum.

<u>Regge poles</u>

$$f_\ell(t) = \frac{r(t)}{\ell - \alpha(t)}$$

residue ← $r(t)$

Regge trajectory ← $\alpha(t)$

Unlike poles of Feynman diagrams

$$\sim \frac{1}{t - m_i^2}$$

Regge poles are moving, $\underline{\alpha = \alpha(t)}$

Before going to the practical examples, recall that scattering amplitudes in QFT possess definite symmetries

$$z = \cos\theta_t = 1 + \frac{2s}{t-4m^2} = \frac{s-u}{t-4m^2}$$

$$(s+u+t = 4m^2)$$

For identical particles in a scattering process amplitudes obey crossing symmetry, eg. $s \leftrightarrow u$

Generally, $A(s,t) = A^+(s,t) + A^-(s,t)$

$$A^{\pm}(s,t) = A(s,t) \pm A(u,t)$$

Because $s \leftrightarrow u$ means $z \leftrightarrow -z$, different reaction amplitudes will have different crossing behavior

$$z \leftrightarrow -z$$

Inserting $f_e^{\pm}(t) = \dfrac{r^{\pm}(t)}{e - \alpha^{\pm}(t)}$ into S-W integral:

$$A^{\pm}(s,t) = -\frac{\pi}{2} r^{\pm}(t) \frac{2\alpha^{\pm}+1}{\sin(\pi\alpha^{\pm})} \left[P_\alpha(-z) \pm P_\alpha(z) \right]$$

The factor $\dfrac{1}{\sin\pi\alpha^{\pm}}$ will generate poles at integer values of $\alpha^{\pm}$!

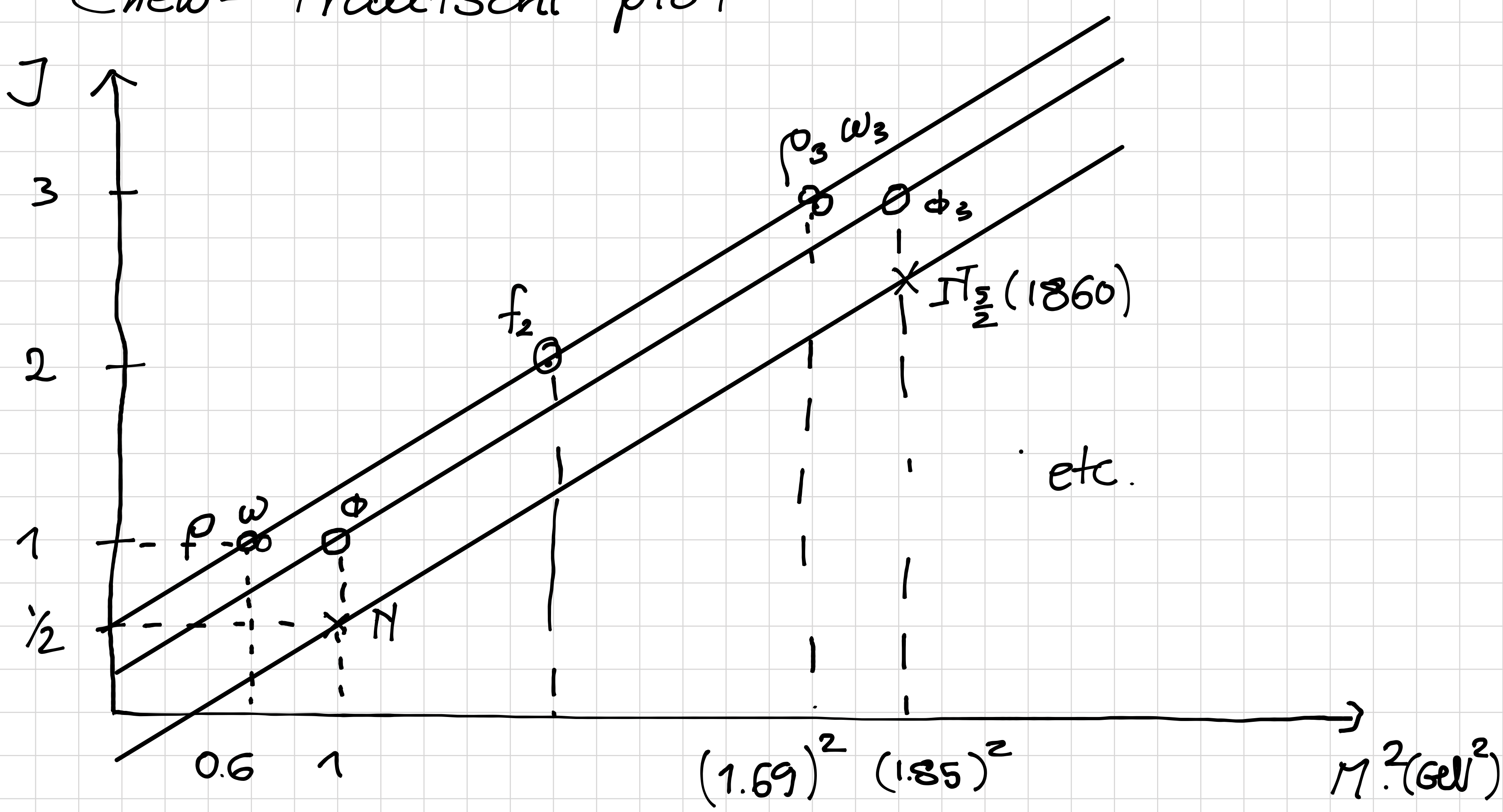
What is the meaning of these poles?

What is the functional form of α?

Examining the spectrum of mesons it was realized that mesons of spin J and mass M

follow the same pattern $J = J_0 + \alpha' M^2$
with $\alpha' \sim 0.9\ \mathrm{GeV}^{-2}$ a universal slope
The same slope characterizes baryon spectrum, too!

Chew–Frautschi plot



Strange mesons and baryons, Δ-baryons all align along the Regge trajectories with a universal slope!

$$\alpha(t) = J_0 + \alpha'(t - M_0^2)$$

J_0 and M_0 are the spin and mass of the lowest realization of a given trajectory

E.g. ρ-trajectory:

lowest member – ρ (775 MeV) with spin 1
next member – ρ_3 (1690 MeV) spin 3
. . .

ω - trajectory: ω (782 MeV) $J_0 = 1$

ω_3 (1670 MeV) $J = 3$

f_2 - trajectory f_2 (1270 MeV) $J_0 = 2$

f_4 (2050 MeV) $J = 4$

f_6 (2510 MeV)

K^* - trajectory K^*(892 MeV) $J_0 = 1$

K_3^*(1780 MeV) $J = 3$

K_5^*(2380 MeV) $J = 5$

Universality of the slope is approximate, but the consistency across the hadron spectrum is absolutely stunning!

Additionally: other quantum numbers
spin; isospin; strangeness;
C, P, G - parity

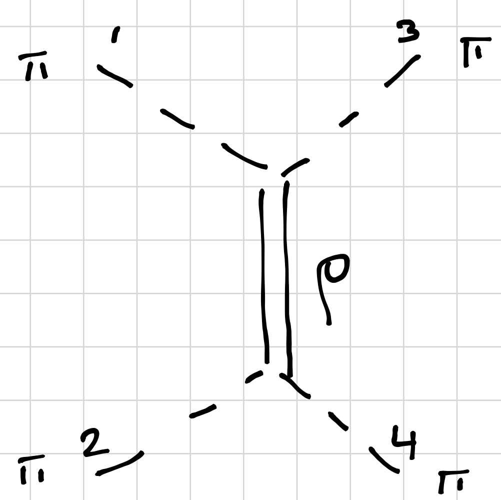
Regge theory: high-energy behavior of scattering amplitudes is governed by t-chan. exchanges of trajectories with appropriate quantum numbers

Depending on the object exchanged, the propagator: $(s-u \equiv 4mV)$

$$z = \frac{s-u}{t-4m^2}$$

$$\sim \frac{1}{\sin[\pi\alpha(t)]}\left[P_\alpha(-z) \pm P_\alpha(z)\right]$$

The high-z behavior governed by the highest power, i.e. $P_\alpha(\pm z) \sim (\pm z)^{\alpha(t)}$



Simple ρ-exchange:

$$\sim \Gamma^{13\mu}_{\rho\pi\pi}(t)\,\frac{1}{t-m_\rho^2}\,\Gamma^{24\nu}_{\rho\pi\pi}(t)\,(-g_{\mu\nu})$$

Regge ρ-exchange:

$$-\left(\frac{\nu}{\nu_0}\right)^{\alpha_\rho(t)-1}\frac{\pi\alpha'}{\sin[\pi\alpha(t)]}\,\frac{\mathcal{S}+e^{-i\pi\alpha(t)}}{2}\,\frac{1}{\Gamma(\alpha(t))}$$

$$\cdot\ \Gamma^{13\mu}_{\rho\pi\pi}(t)\ \Gamma^{24\nu}_{\rho\pi\pi}(t)\cdot(-g_{\mu\nu})$$

The signature $\mathcal{S}=(-1)^J$

$\frac{1}{\Gamma(\alpha(t))}$ has zeros for $\alpha(t)=0,-1,-2,\dots$

$\to$ cancels poles of $\frac{1}{\sin(\pi\alpha)}$, so that only poles at positive integer t remain

$$\frac{S+e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)} = \frac{1}{2}\left[\frac{S+\cos\pi\alpha}{\sin\pi\alpha} - i\right]$$

$$S=+1 \to \frac{1}{2}\left[\cot\frac{\pi\alpha}{2} - i\right] \to \text{poles for } \alpha=0,2,\dots$$

$$S=-1 \to \frac{1}{2}\left[\operatorname{tg}\frac{\pi\alpha}{2} - i\right] \to \text{poles for } \alpha=1,3,\dots$$

For $t \to M_\rho^2$ $\quad \alpha_\rho(t) = 1 + \alpha'(t-M_\rho^2)$

$$\hookrightarrow \left(\frac{\nu}{\nu_0}\right)^{\alpha'(t-M_\rho^2)}\left(-\frac{\pi\alpha'}{2}\right)\left[\frac{\sin\left[\frac{\pi}{2}+\frac{\pi\alpha'}{2}(t-M_\rho^2)\right]}{\cos\left[\frac{\pi}{2}+\frac{\pi\alpha'}{2}(t-M_\rho^2)\right]} - i\right]\frac{1}{\Gamma(1+\varepsilon)}$$

$$= +\frac{\pi\alpha'}{2}\frac{1}{\sin\frac{\pi\alpha'}{2}(t-M_\rho^2)} = \frac{1}{t-M_\rho^2} + \text{finite}$$

↳ in the vicinity of the physical t-channel pole recover the Feynman propagator

Scattering amplitudes in the s-channel kinematics ($s \gg$, $t<0$) know the content of the amplitude in the t-channel kinematics ($t>0$) and its leading singularities (pole closest to $t=0$)

At the same time, "Reggeon" exchange — exchange of a particle of variable spin $\alpha(t)$

The energy dependence governed by

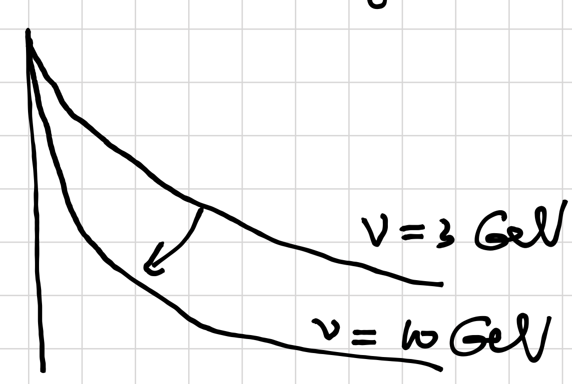
$\left(\frac{\nu}{\nu_0}\right)^{\alpha(t)}$ $\quad$ ($\left(\frac{\nu}{\nu_0}\right)^{-1}$ is compensated by the structure of the ampl., e.g. for spin-1 exchange it is $(p_1+p_3)_\mu (p_2+p_4)^\mu \sim \nu$)

Exchange of higher spin with Feynman propagators leads to unphysical HE behavior ν^J $\longrightarrow$ cured by Regge propagator.

$$\left(\frac{\nu}{\nu_0}\right)^{\alpha(t)} = e^{\alpha(t)\ln\left(\frac{\nu}{\nu_0}\right)}$$

– exponential form factor for $t<0$ drops

Shrinkage of the forward peak of $\frac{d\sigma}{dt}$

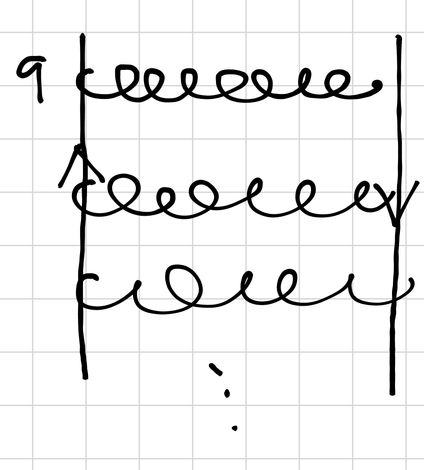


For leading trajectories $\longrightarrow$ $\left.\frac{d\sigma}{dt}\right|_{t=0}$ grows with ν

The slope of $\frac{d\sigma}{dt}$ $\sim$ interaction radius

$\longrightarrow$ growing interaction radius

In perturbative QCD language a reggeon exchange $\sim$ gluon ladders

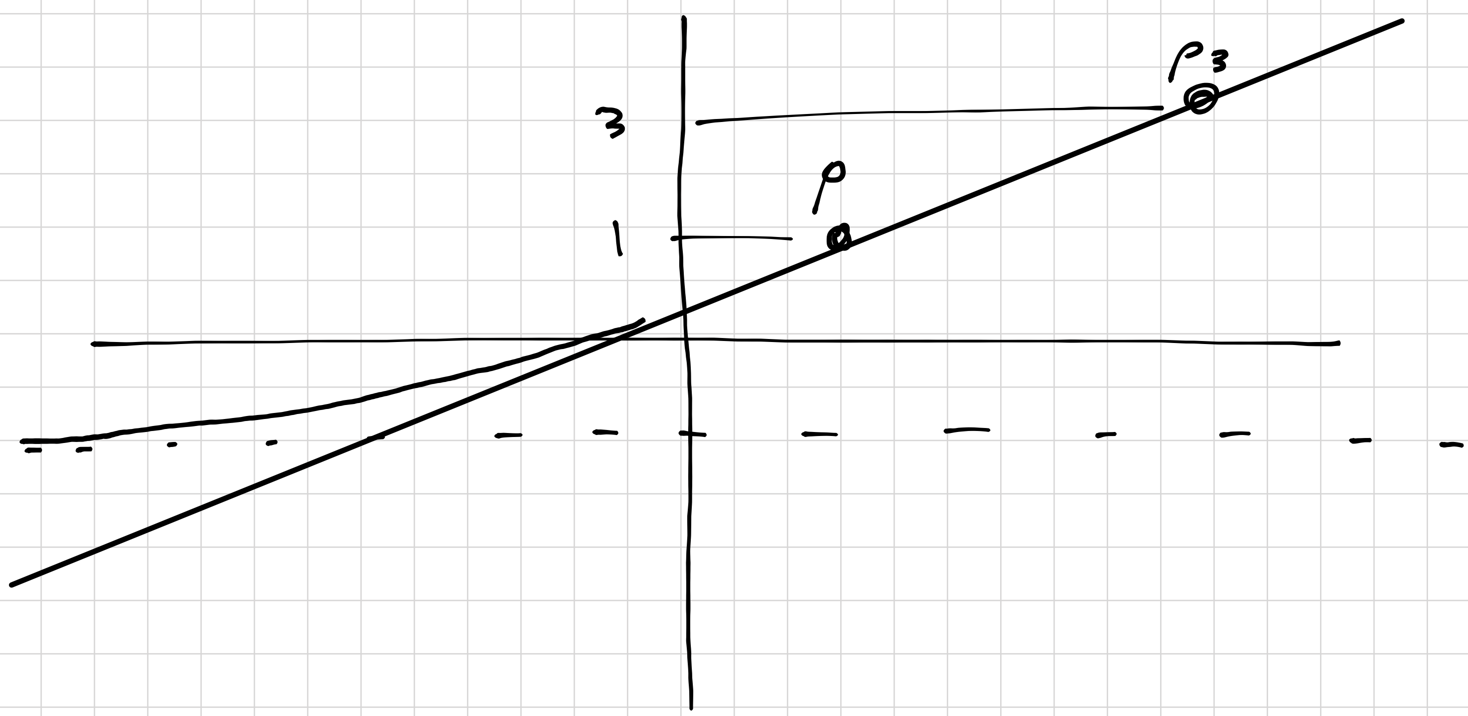
$\sim$ 

accompanying a $q\bar{q}$ – exchange

$q\bar{q}$ (in colorless state) convey quantum numbers

Strictly speaking, we only know that $\alpha(t)$ is linear at $t>0$ (linear trajectories from hadron spectra) and small negative t (Regge exchanges do describe scattering data!)

But what happens at large negative t?



QCD: hard quark exchange $\sim 1/t \sim \frac{1}{s}$

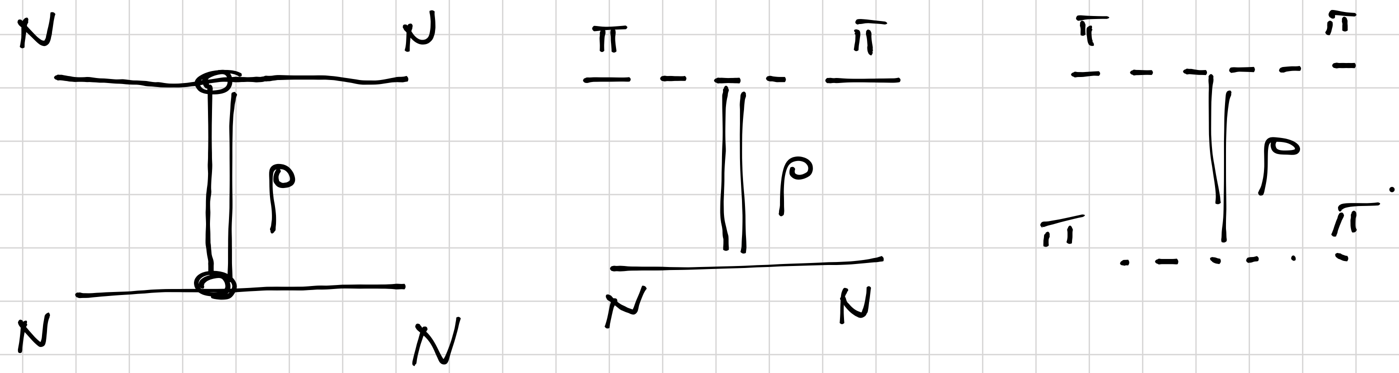
$s \sim |t| \gg$

$\hookrightarrow$ expect $\alpha(t) \to$ const.

(α "saturates")

There might be a change of trend at some value of t $\to$ "saturated" Regge trajectory

Regge calculus $\rightarrow$ just like Feynman rules



$(\Gamma_{\rho NN})^2$ $\qquad$ $\Gamma_{\rho NN}\,\Gamma_{\rho\pi\pi}$ $\qquad$ $(\Gamma_{\rho\pi\pi})^2$

$\hookrightarrow$ $A^{\rho}_{NN} \cdot A^{\rho}_{\pi\pi} = (A^{\rho}_{\pi N})^2$ $\rightarrow$ Regge factorization

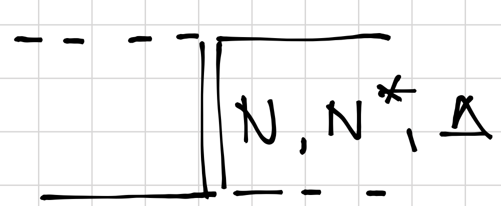
Vector exchanges $J^P = 1^-, 3^- \ldots$ natural parity

Axial-v. $J^{PC} = 1^+, 3^+$ unnatural parity

Scalar $J^P = 0^+, 2^+, \ldots$ natural

$J^P = 0^-, 2^-, \ldots$ unnatural

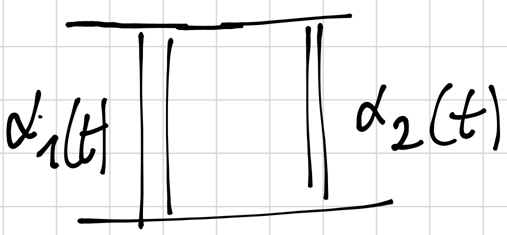
Similarly $\hookrightarrow$ baryon exchanges



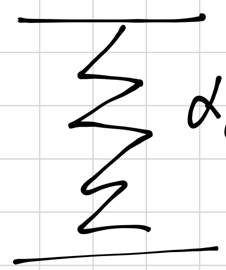
$\rightarrow$ lead to a backward peak (small u)

meson exchanges $\rightarrow$ forward peak (small t)

One can also compute rescattering with Regge exchanges $\longrightarrow$ Regge cuts



$\longrightarrow$



$$\alpha_{c_{12}}(t) = \alpha_{c_{12}}(0) + \alpha'_{c_{12}} t$$

$$\alpha_{c_{12}}(0) = \alpha_1(0) + \alpha_2(0) - 1 \qquad \alpha'_{c_{12}} = \frac{\alpha'_1 \alpha'_2}{\alpha'_1 + \alpha'_2}$$

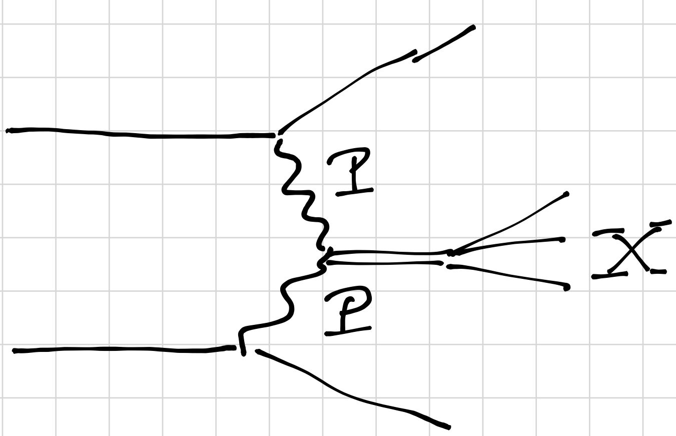
from phase space $\longrightarrow$ -1

One can also write down reggeon - reggeon scattering amplitudes, e.g. pomeron - pomeron fusion in central - rapidity production

(pseudo) rapidity $\eta = -\ln\left(\tan\frac{\theta}{2}\right)$

Collider: $\xrightarrow{p_1}$ $\qquad\qquad \xleftarrow{p_2}$

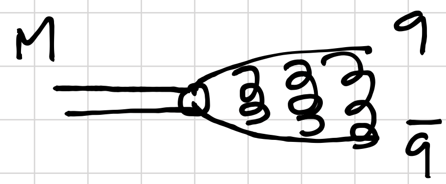
$\eta = \infty \qquad\qquad \eta = -\infty$



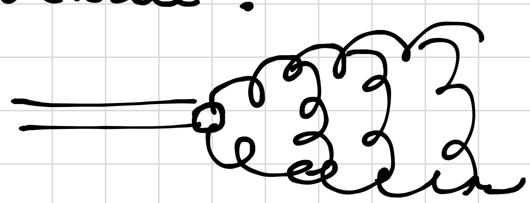
Does pomeron exchange correspond to a real meson state?

Data: $\alpha_P(t) \simeq 1.09 + 0.25\, t/\mathrm{GeV}^2$

Meson:



Pomeron: Glueball?



Quantum numbers 2^{++} possible!

Would have to lie around $M_{2^{++}} \sim 2$ GeV

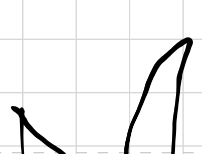
↳ Lattice QCD simulations

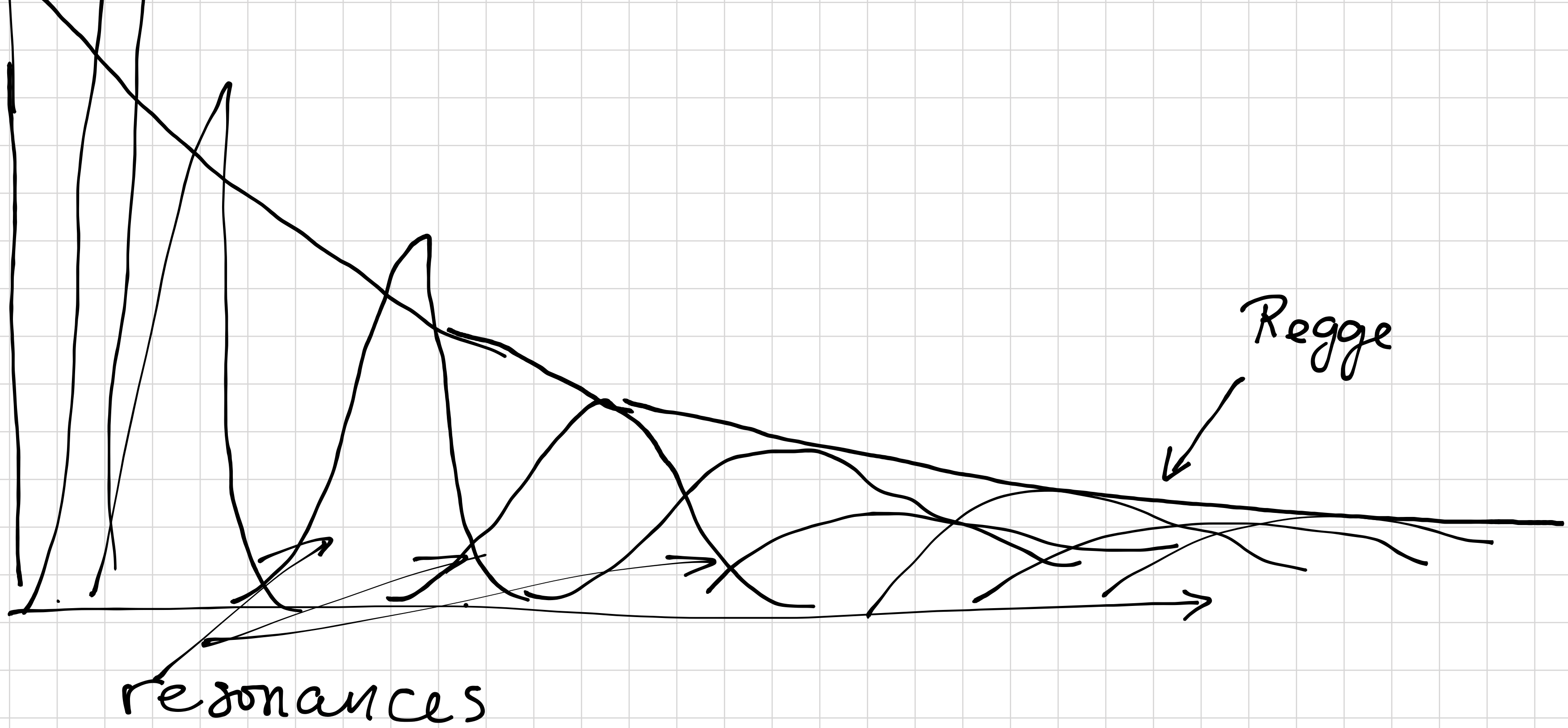
At present no particle could be clearly identified with a glueball.

Regge theory and duality

Regge exchanges determine s-channel scattering at high energy;
via Regge trajectory informed about the spectrum in the t-channel

From the s-channel perspective:
some s-channel resonances, isolated at low energy, overlapping at high energy





At high enough energy resonances cannot be distinguish any more $\longrightarrow$ Regge description more effective. Regge amplitude can be continued into resonance region.

The strength of s-channel resonances is equal to that of t-channel Regge exch. on average (a weighted $\int$ over energy).

Because Regge = sum over the t-chan poles = resonances, it implies

$$A(s,t) = \sum_{res.\,s} A(s,t) = \sum_{res.\,t} A(s,t)$$

The full amplitude knows its spectrum in both s and t channels.

Duality and crossing are directly incorporated in Veneziano Model for $\pi\pi$ - scatter

$$A(s,t,u) = \frac{\beta}{\pi}\Big[B\big(1-\alpha(t), 1-\alpha(s)\big)$$
$$+ \text{—''—}\ (t,u) + \text{—''—}\ (s,u)\Big]$$

Euler β - function

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

This amplitude is fully symmetric $(s \leftrightarrow t \leftrightarrow u)$ and has Regge behavior in every channel, e.g.

$$A_{s\to\infty} \sim -\beta(t)\,\mathrm{ctg}\,\alpha(s)\,[\alpha(s)]^{\alpha(t)-1}$$

Moreover, for linear trajectories it is enough to require

$$\alpha(s) + \alpha(t) + \alpha(u) = 2$$

Then

$$A = \frac{\beta}{\pi^2}\,\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(u))$$
$$\cdot\big[\sin(\pi\alpha(s)) + \sin(\pi\alpha(t)) + \sin(\pi\alpha(u))\big]$$

This amplitude is built of resonances (poles of the Γ-fn) in each channel and is automatically Regge-behaved in every channel under corresponding conditions.

The Veneziano model is beautiful but

has its problems (does not describe $\pi\pi$ scattering) but can be massaged to be made compatible;
If implementing pomeron ($\alpha(0)=1$) the model contains tachyons (poles occur at imaginary mass).

The only dimensionful parameter is the slope α' that is universal.

Comparing the spectrum of a spinning string with constant tension $E \propto L$, as $M^2 \propto J$

This motivated study of string theory as a fundamental theory in the 1970's.