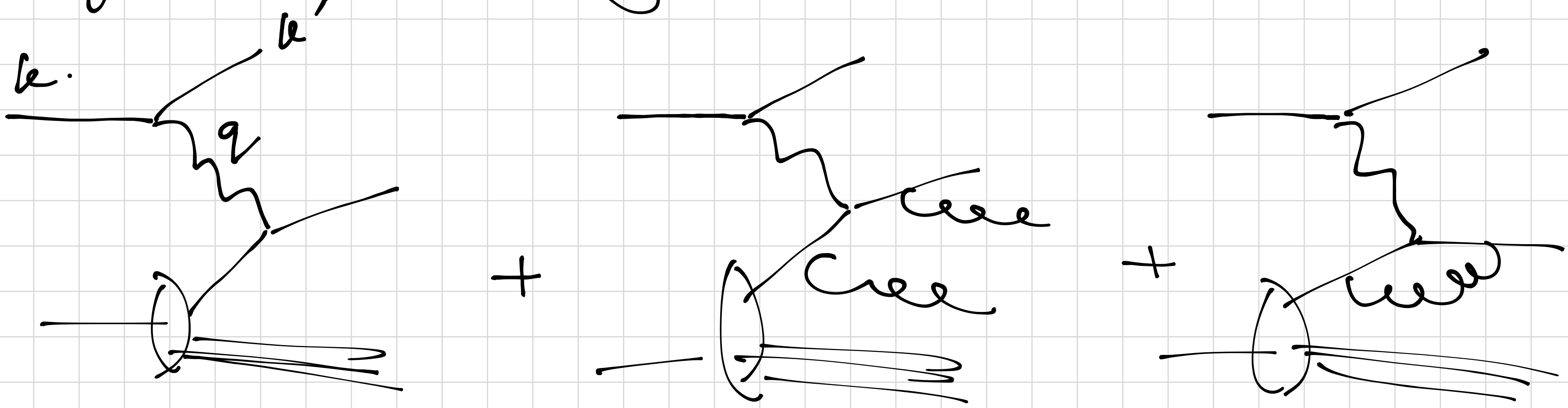
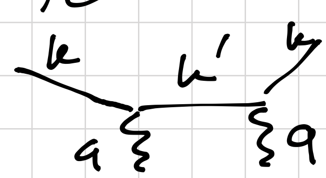
Lecture 10

Deep inelastic scattering
Bjorken scaling and violation thereof



$$d\sigma_{ep\to eX} = \frac{\alpha^2}{\pi}\frac{1}{Q^4}\frac{d^3\vec{k}'}{(pk)E'} L_{\mu\nu} W^{\mu\nu}$$

Leptonic tensor



$$L^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\, \not{k}\gamma^\mu \not{k}'\gamma^\nu = 2\left[k^\mu k'^\nu + k^\nu k'^\mu - (kk')g^{\mu\nu}\right]$$

$$W^{\mu\nu} = \frac{1}{2\pi}\sum_X \langle p|j^\mu(0)|X\rangle\langle X|j^\nu(0)|p\rangle (2\pi)^4\delta^4(p+q-p_X)$$

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x,Q^2)$$
$$+ \frac{1}{(pq)}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right) F_2(x_B,Q^2)$$

$$x_B = \frac{Q^2}{2pq}\ ; \quad q^2 = -Q^2 < 0\ ; \quad \nu = \frac{(pq)}{M} = \frac{Q^2}{2Mx_B}$$

Operator product expansion (OPE)

Typical object of study

$$T\{J^\mu(y)\, J^\nu(0)\}\Big|_{y\to 0} = \sum_n C_n(y)\, O_n^{\mu\nu}(0)$$

$$T^{\mu\nu} = i\int d^4y\, e^{iqy} \langle P|\, T\{J^\mu(y)\, J^\nu(0)\}\,|P\rangle$$

Importantly: OPE holds on the <u>operator</u> level!

This means: <u>Wilson coefficients</u> C_n are process independent, the only place where process dependence appears are the m.el. of the operator $O_n^{\mu\nu}$, e.g. $\langle P|\, O_n^{\mu\nu}\,|P\rangle$

We want to compute Wilson coefficients by evaluating $T^{\mu\nu}$ on quark state $|p\rangle$

$$t^{\mu\nu} = i\int d^4y\, e^{iqy} \langle p|\, T\{J^\mu(y)\, J^\nu(0)\}\,|p\rangle$$

$$= -e_q^2\left[\bar{u}(p)\frac{\gamma^\mu(\not{p}+\not{q})\gamma^\nu}{(p+q)^2+i\varepsilon}u(p) + \bar{u}(p)\frac{\gamma^\nu(\not{p}-\not{q})\gamma^\mu}{(p-q)^2+i\varepsilon}u(p)\right]$$

Compute the spin-independ part of the Compton tensor; use $p^2 \simeq 0$, $\not{p}\, u(p) = 0$

Expand the propagator

$$-\frac{1}{(p+q)^2} = \frac{1}{Q^2 - 2pq} = \frac{1}{Q^2}\sum_n \left(\frac{2pq}{Q^2}\right)^n$$

$\hookrightarrow \quad \frac{1}{Q^2}\,\bar{u}(p)\gamma^\mu(\not{p}+\not{q})\gamma^\nu u(p) \sum_n \left(\frac{2pq}{Q^2}\right)^n$

Now : every power of $p^{\mu_i} \longrightarrow$ from $i\partial^{\mu_i}$ acting on the external state of an operator $O^{\mu\nu\mu_i\ldots\mu_i}$

every power of $q^\mu \longrightarrow$ move to Wilson coeff. $C^{\mu_i\ldots\mu_j}$.

$$\bar{u}(p)\gamma^\mu(\not{p}+\not{q})\gamma^\nu u(p) = \bar{u}(p)\left[\gamma^\mu q^\nu + \gamma^\nu q^\mu - g^{\mu\nu}\not{q} + \gamma^\mu p^\nu + \gamma^\nu p^\mu\right] u(p)$$

Second term

$$\hookrightarrow \frac{1}{Q^2}\,\bar{u}(p)\left[-\left(\gamma^\mu q^\nu + \gamma^\nu q^\mu - g^{\mu\nu}\not{q}\right) + \gamma^\mu p^\nu + \gamma^\nu p^\mu\right] u(p)$$

$$\cdot \sum_n \left(\frac{-2pq}{Q^2}\right)^n$$

Putting together:

$$t_q^{\mu\nu} = e_q^2\Bigg[\frac{2}{Q^2}\sum_{n=0,2,\ldots} \underbrace{\bar{\Psi}_q(x)\left(\gamma^\mu i\partial^\nu + \gamma^\nu i\partial^\mu\right) i\partial^{\mu_1}\ldots i\partial^{\mu_n}\Psi_q(x)}_{O_{n+2}^{\mu\nu\mu_1\ldots\mu_n}} \times \left(\frac{2}{Q^2}\right)^n q_{\mu_1}\ldots q_{\mu_n}$$

$$+ \frac{2}{Q^2}\sum_{n=1,3\ldots} \bar{\Psi}_q(x)\left(\gamma^\mu q^\nu + \gamma^\nu q^\mu - g^{\mu\nu}\not{q}\right) i\partial^{\mu_1}\ldots i\partial^{\mu_n}\Psi_q(x) \times \left(\frac{2}{Q^2}\right)^n q_{\mu_1}\ldots q_{\mu_n}\Bigg]$$

Basis of gauge-invariant operators that transform as irreducible representation of Lorentz group and have spin s are symmetric and traceless :

$$\hat{O}_{s,r}^{\mu_1 \dots \mu_s} = \bar{\psi} \gamma^{\mu_1} i\partial^{\mu_2} \dots i\partial^{\mu_s} (-\Box)^r \psi$$

$$+ \text{ symmetrization } - \text{ traces}$$

Mass dimension of such an operator is

$$d = 2 + s + 2r$$

Introduce twist $t \equiv d - s = 2 + 2r$

Because each $\Box$ will pick the small $p^2 \ll Q^2$, the lowest $r=0$ will dominate $\rightarrow$ twist 2 is the lowest twist; higher twists will always be accompanied by inverse powers of Q^2

Example : spin-2, twist-2 operator

$$\hat{O}_{2,0}^{\mu\nu} = \bar{\psi}(x) \left(\gamma^\mu i\partial^\nu + \gamma^\nu i\partial^\mu - \frac{1}{2} g^{\mu\nu} i\partial\!\!\!/ \right) \psi(x)$$

Finally, gauge invariance w.r.t. strong int.

$$i\partial^\mu \longrightarrow iD^\mu$$

Canonical basis for gauge-inv. twist-2 quark ops.

$$\hat{O}_q^{\mu_1 \dots \mu_n}(x) = \bar{\psi}_q(x) \gamma^{\mu_1} iD^{\mu_2} \dots iD^{\mu_n} \psi_q(x)$$

$$+ \text{ symmetrization } - \text{ traces}$$

In terms of these operators,

$$i \int d^4x \, e^{iqx} T\{ J^\mu(x) J^\nu(0) \} = \sum_q e_q^2$$

$$\times \sum_{n=2,4\dots}^{\infty} \frac{(2q_{\mu_1}) \dots (2q_{\mu_n})}{Q^{2n}} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \hat{O}_q^{\mu_1 \dots \mu_n}$$

$$+\sum_{n=2,4\ldots} \frac{(2q_{\mu_3})\ldots(2q_{\mu_n})}{Q^{2n-2}} \left(g^{\mu}{}_{\mu_1} - \frac{q^{\mu}q_{\mu_1}}{q^2}\right)\left(g^{\nu}{}_{\mu_2} - \frac{q^{\nu}q_{\mu_2}}{q^2}\right) \cdot \hat{O}_q^{\mu_1\ldots\mu_n} \Bigg\}$$

Detailed derivation $\longrightarrow$ exercise

Now connect to DIS $\longrightarrow$ insert $\hat{O}_q^{\mu_1\ldots\mu_n}$ between proton states

Spin-independent part:

$$\sum_{\text{spins}} \langle P | \hat{O}_q^{\mu_1\ldots\mu_n} | P \rangle = A_q^n P^{\mu_1}\ldots P^{\mu_n} - \text{traces}$$

$$\Rightarrow T^{\mu\nu} = \sum_q e_q^2 \Bigg\{ \left(-g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2}\right) \sum_{n=2,4\ldots} \left(\frac{1}{x_B}\right)^n A_q^n$$

$$+ \left(P^{\mu} - \frac{Pq}{q^2}q^{\mu}\right)\left(P^{\nu} - \frac{Pq}{q^2}q^{\nu}\right) \sum_{n=2,4\ldots} \left(\frac{1}{x_B}\right)^n A_q^n \cdot 4\frac{x_B^2}{Q^2} \Bigg\}$$

Since $T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2}\right) T_1 + \frac{1}{Pq} \tilde{P}^{\mu}\tilde{P}^{\nu} T_2$

$$T_1 = \frac{1}{2x_B} T_2 = \sum_q e_q^2 \sum_{n=2,4\ldots}^{\infty} \left(\frac{1}{x_B}\right)^n A_q^n$$

Structure functions $F_i = \frac{1}{2\pi} \operatorname{Im} T_i$

$$F_1 = \frac{1}{2}\sum_q e_q^2\, q(x) \Rightarrow q(x) = \frac{1}{\pi} \sum_{n=2\ldots} \frac{1}{x^n} \operatorname{Im} A_q^n$$

Operator definition of quark PDF $q(x)$

What are the coeffs. A_q^n ?

Introduce Mellin moments of PDF

$$C_q^m = \int_0^1 dx\, x^{m-1} q(x)$$

$$\omega = \frac{1}{x}$$

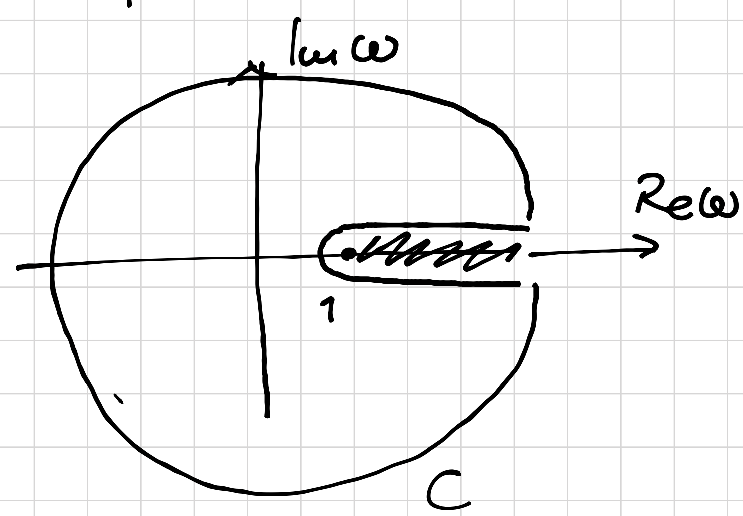
$$q(x) = \frac{1}{\pi} \sum \frac{1}{x^n} \operatorname{Im} A_q^n = \frac{1}{\pi} \sum \omega^n \operatorname{Im} A_q^n$$

$$C_q^m = \frac{1}{\pi} \operatorname{Im} \int_1^\infty d\omega \sum_n \omega^{n-m-1} A_q^n$$

Analytical properties of A_q^n

Complete to a closed contour

Inside the contour A_q^n is analytic



$$C_q^m = \frac{1}{2\pi i} \oint_C d\omega \sum_n \omega^{n-m-1} A_q^n$$

↪ Then, only a simple pole in ω survives

$$\Rightarrow C_q^m = A_q^m$$

$q(x)$ is defined in OPE in terms of its Mellin moments

When computing $\langle P | \hat{O}_q | P \rangle$ we neglected the traces that are required to be there to ensure that $\hat{O}_q$ transforms as irreducible spin-d2 tensor (trace removes the spin-0 component).

To include them:

Nachtmann variable

$$\xi = \frac{2x}{1+\sqrt{1+\frac{4M^2x^2}{Q^2}}} \equiv \frac{2x}{1+r} \Bigg|_{\frac{4M^2x^2}{Q^2} \to 0} \to x$$

In presence of TMC (target mass corrections) the nice equality $C_q^m = A_q^m$ becomes an infinite series with terms $\sim \left(\frac{M^2}{Q^2}\right)^n$ and higher Mellin moments. To bring back the 1-1 correspondence

Nachtmann moments, e.g.

$$\mu_2^n(Q^2) = \int_0^1 dx \, \frac{\xi^{n+1}}{x^3} \left[\frac{3+3(n+1)r+n(n+2)r^2}{(n+2)(n+3)} \right] F_2^{TMC}(x,Q^2)$$

$$r \to 1 \Rightarrow [\ldots] \to \frac{3+3(n+1)+n(n+2)}{(n+2)(n+3)} = 1$$

$$\frac{\xi^{n+1}}{x^3} \to x^{n-2}$$

$$M_2^n(Q^2) = \int_0^1 dx \, x^{n-2} F_2 \qquad \text{Mellin moment}$$

TMC are also called kinematical higher twist

Higher twist $\Rightarrow$ leads to corrections $\sim \frac{1}{Q^{2n}}$

Here one distinguishes corrections $\sim \left(\frac{1}{Q^2}\right)^n$ stemming from the twist-2 operators when evaluated between proton states with finite mass.

Sum rules for PDFs

Momentum sum rule $\quad \tilde{O}^{\mu\nu} = \gamma^\mu i D^\nu + \gamma^\nu i D^\mu$

$$\sum_j \langle P | \underbrace{(\gamma^\mu D^\nu + \gamma^\nu D^\mu)}_{\text{Canonical energy-mom. tensor}} | P \rangle = P^\mu P^\nu$$

$$\hookrightarrow \sum_q \int_0^1 dx\, x\, q(x) + \int_0^1 dx\, x\, \underset{\substack{\uparrow \\ \text{gluon PDF}}}{g(x)} = 1$$

Electric charge $\quad \hat{O}^\mu = \gamma^\mu$

$$\langle P | \sum_q e_q \bar{q} \gamma^\mu q | P \rangle =$$

Keep track of $e_{\bar{q}} = -e_q$

$$\hookrightarrow \int_0^1 dx\, (u(x) - \bar{u}(x)) = 2 \quad (\text{in proton})$$

$$\int_0^1 dx\, (d(x) - \bar{d}(x)) = 1$$

$\hookrightarrow$ Related to Gross-Llewellyn-Smith SR:

For inclusive (νp) and $(\bar{\nu} p)$ scattering

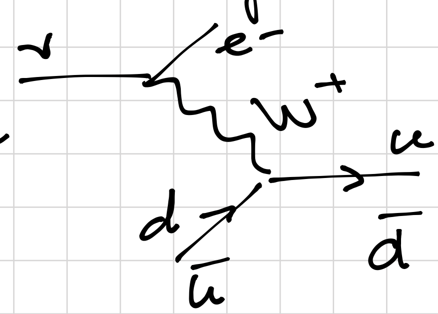
Additionally to F_1 and F_2 there is a Parity-violating SF F_3

$$T^{\mu\nu} = \dots - \frac{i\varepsilon^{\mu\nu\alpha\beta} P_\alpha q_\beta}{2Pq} F_3$$

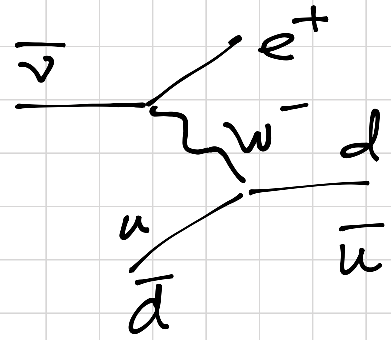
PV $\longrightarrow$ due to interference $AV \times V$
(AV not present in EM interaction)

But particles and antiparticles have opposite sign of V coupling ($C=-1$) and same sign AV coupling ($C=+1$)

$$\hookrightarrow \frac{d\sigma^{\nu p}}{dx\,dy} - \frac{d\sigma^{\bar\nu p}}{dx\,dy} \sim F_3^{\nu p} + F_3^{\bar\nu p}$$

$F_3^{\nu p} \sim$


are not the same

$F_3^{\bar\nu p} \sim$


$$\frac{1}{2}\int_0^1 dx \left(F_3^{\nu p} + F_3^{\bar\nu p}\right) = \int_0^1 dx \left[u - \bar{u} + d - \bar{d}\right] = 3$$

Gottfried SR

$$\int_0^1 \frac{dx}{x}\left(F_2^{ep} - F_2^{en}\right) = \frac{1}{3} \qquad \text{etc.}$$