Script — Lecture 1 — Intro

QFT — Basic object → field $\hat{\varphi}(x,p)$
Operator-valued fn. of space-time, momenta...

Goal: join QM (operator, unitarity) with special relativity

Equations, observables are unchanged upon applying Lorentz transformation

L.T. def. $\Lambda^{\mu\nu}\Lambda^{\alpha\beta}\cdot g_{\nu\beta} = g^{\mu\alpha} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

metric

Rotations in 3D + Boosts

Rotation by θ around z-axis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boost in z-direction (v in units of c)

$$\begin{pmatrix} \cosh\beta & & & \sinh\beta \\ & 1 & & \\ & & 1 & \\ \sinh\beta & & & \cosh\beta \end{pmatrix}$$

$$\cosh\beta = \frac{1}{\sqrt{1-v^2}}$$

$$\sinh\beta = \frac{\pm v}{\sqrt{1-v^2}}$$

$$\Lambda : x^\mu \longrightarrow x'^\mu = \Lambda^{\mu\alpha} x_\alpha$$

Scalar field $\varphi(x^\mu) \to \varphi' = \varphi(\Lambda^{-1\,\mu\alpha} x_\alpha)$
(active transformation)

Vector $A^\mu \to \Lambda^{\mu\nu} A_\nu$

Tensor $T^{\mu_1 \dots \mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \dots \nu_n}$

$A_\mu B^\mu \to$ Lorentz scalar
(paired indices stay intact; unpaired indices transform)

Position not a dynamical variable $\to$ index of the field $\varphi(\vec{x}, t)$

Lagrangian density $\to$ Lorentz scalar

KG: $\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2} m^2 \varphi$

Euler-Lagrange: $\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0$

$\hookrightarrow (\partial^2 + m^2)\varphi = 0$ KG equation

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\hookrightarrow \partial_\mu F^{\mu\nu} = 0$ Maxwell Eq. etc.

Noether theorem: each <u>continuous</u> symmetry $\longleftrightarrow$ conserved current

$\varphi_a \to \varphi_a + \delta\varphi_a \Rightarrow \mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, $\delta\mathcal{L} = \partial_\mu J^\mu$

$$\hookrightarrow \partial_\mu \underbrace{\left(\delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - J^\mu \right)}_{\text{conserved current}} = 0$$

*) Translation (infinitesimal)

$$x^\mu \to x^\mu - \varepsilon^\mu \Rightarrow \phi_a(x) \to \phi_a + \varepsilon^\nu \partial_\nu \phi_a$$

$$\mathcal{L} \to \mathcal{L} + \underbrace{\varepsilon^\nu \partial_\nu \mathcal{L}}_{\delta \mathcal{L}} \Rightarrow \text{Conserved current } T^{\mu\nu}$$

Energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial^\nu \varphi_a - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = 0 \to 4 \text{ conserved comp.}$$

$$E = \int d^3x\, T^{00} \qquad \vec{P}_i = \int d^3x\, T^{0i}$$

KG Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2$

$$\to T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

$$E = \int d^3\vec{x} \left[\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right]$$

$$\vec{P} = \int d^3\vec{x}\, \dot{\varphi} \vec{\nabla} \varphi$$

Inf. Lorentz transf. $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ $\qquad \omega^{\mu\nu} + \omega^{\nu\mu} = 0$

$\mathcal{L}$ invariant $\Rightarrow$ $J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}$

conserved $\partial_\mu J^{\mu\alpha\beta} = 0$

(Exercise: Find all conserved currents of K-G field)

Hamilton formalism $\Pi_a(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}_a}$

conjugate mom. to φ_a

$\rightarrow$ Hamiltonian density $\mathcal{H} = \Pi_a \dot{\varphi}_a - \mathcal{L}$

Hamiltonian $H = \int d^3x \, \mathcal{H}$

Real scalar: $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} V(\varphi)$

$$\mathcal{H} = \frac{1}{2}\left(\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2 + V(\varphi)\right)$$

$\rightarrow$ total energy from time translation inv.

To solve it we need to diagonalize $\mathcal{H}$

Fourier transform $\varphi(\vec{x},t) = \int d^3\vec{p} \, e^{-i\vec{p}\vec{x}} \varphi(\vec{p},t)$

$$\left(\partial_t^2 - \vec{\nabla}^2 + m^2\right)\varphi(\vec{x},t) = 0 \Rightarrow \left(\partial_t^2 + (\vec{p}^2 + m^2)\right)\varphi(\vec{p},t) = 0$$

This is just H.O.

To quantize real scalar field: need to quantize an infinite # of H.O.

How is H.O. quantized?

QM: $\hat{q}$, conjugate momentum $\hat{p}$: $[\hat{q}, \hat{p}] = i$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$$

Ladder op's. $\hat{a} = \sqrt{\frac{m\omega}{2}}\,\hat{q} + \frac{i}{\sqrt{2m\omega}}\,\hat{p}$

$$\hat{a}^+ = \text{—''—} - \text{—''—}$$

$$[a^+, a] = 1$$

$$\Rightarrow \quad H = \omega\left(a^+ a + \tfrac{1}{2}\right) \qquad [a, H] = \omega a$$

$$[a^+, H] = -\omega a^+$$

$$i \frac{d}{dt} a = [a, H] \longrightarrow \quad a(t) = a(0)\, e^{-i\omega t}$$

$$a^+(t) = a^+(0)\, e^{i\omega t}$$

We can determine the spectrum

$$H\,|n\rangle = E_n\,|n\rangle \qquad H|0\rangle = \frac{\omega}{2}|0\rangle \equiv E_0\,|0\rangle$$

n-part. state $\qquad (H - E_0)\,|n\rangle = n\omega\,|n\rangle$

$$|n\rangle = \frac{1}{\sqrt{(n+1)!}}\,(a^+)^n\,|0\rangle$$

n particles in the mode with ω_p

Many particles q_a, p_b $\quad [\hat{q}_a, \hat{p}_b] = i\delta_{ab}$

$$[\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0$$

Field φ_a, conj. mom π_b

$$[\varphi_a(\vec{x}, t), \pi_b(\vec{y}, t)] = i\,\delta^3(\vec{x} - \vec{y})\,\delta_{ab}$$ Equal time

$$\varphi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}}\left[\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}\right]$$

$$\pi = \dot{\varphi} = -i \int \frac{d^3\vec{p}}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left[\hat{a}_p e^{-ipx} - \hat{a}_p^+ e^{ipx}\right]$$

$$[\hat{a}_p, \hat{a}_q^+] = (2\pi)^3\,\delta^3(\vec{p} - \vec{q})$$

$$\hookrightarrow H = \int d^3\vec{x}\,\mathcal{H} = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\omega_p\left[a_p^+ a_p + \tfrac{1}{2}(2\pi)^3\delta^3(0)\right]$$

$\underset{\circ}{?}$ normal ordering : all $a^\dagger$ to the left (creation)

all a to the right (annihilation)

Why? $\to$ $\underline{a|0\rangle = 0}$

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p \Big[\hat{a}_p^\dagger \hat{a}_p + \underbrace{\tfrac{1}{2}(2\pi)^3 \delta^3(0)}_{=\infty}\Big]$$

(inf. Volume

Vacuum energy

$$H|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p\, \underbrace{\tfrac{1}{2}(2\pi)^3 \delta^3(0)}_{=\infty} = E_0|0\rangle$$

$$(2\pi)^3\delta^3(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3\vec{x}\, e^{i\vec{p}\vec{x}}\Big|_{p=0} = V$$

Energy density $\quad \mathcal{E}_0 = \dfrac{E_0}{V}$

$\mathcal{E}_0$ also infinite $\quad \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p = \infty$

Remove vac. energy

$$\hat{H}_{KG} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p\, \hat{a}_p^\dagger \hat{a}_p$$

EM tensor $\quad T^{\mu\nu} = \partial^\mu\varphi\, \partial^\nu\varphi - g^{\mu\nu}\mathcal{L}$

$\hat{H} = \int d^3\vec{x}\, T^{00}$ $\hat{P}^i = \int d^3\vec{x}\, T^{0i}$

conserved currents (Noether theorem)

Construct Fock space

1 part. $\longrightarrow$ Hilbert space $\mathcal{H}$

n part. $\longrightarrow$ Fock space $\mathcal{F} = \oplus_n \mathcal{H}$

$$|p_1\, p_2 \ldots p_n\rangle = \hat{a}^+_{p_1} \ldots \hat{a}^+_{p_n} |0\rangle$$

Properly normalized 1-p state

$$|p\rangle = \sqrt{2\omega_p}\, \hat{a}^+_p |0\rangle$$

In general: to quantize real scalar field

Lagr. $\longrightarrow$ symmetries $\longrightarrow$ conserved currents

- $\rightarrow$ identify all
- $\rightarrow$ put hats on all
- $\rightarrow$ check that all generators obey commutation relations $[\hat{Q}_\alpha, \hat{Q}_\beta] = i f_{\alpha\beta\gamma} \hat{Q}_\gamma$
- $\rightarrow$ construct Fock space

This defines unitary representation of the Poincare group (Λ + transl.)

Feynman propagator

$$\Delta_F(x-y) = \langle 0 | T\{\varphi(x), \varphi(y)\} | 0 \rangle$$
$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)}$$

F.P. = Green's fn. of KG equation

$$(\partial^2 + m^2) i\Delta_F(x-y) = \delta^{(4)}(x-y)$$

Interactions $H = H_0 + \underbrace{\frac{g^{(n)}}{n!}\varphi^n}_{H_{int}}$ etc.

Perturbation theory $\hat{H} = \hat{H}_0 + \hat{H}_{int}$

t-dep. is Schroed. pict. $i\frac{d}{dt}|\Psi_S\rangle = \hat{H}_S|\Psi_S\rangle$

Interaction picture:

W.F. $|\Psi_I(t)\rangle = e^{i\hat{H}_0 t}|\Psi_S(t)\rangle$

Op. $O_I = e^{i\hat{H}_0 t} O_S e^{-i\hat{H}_0 t}$

$$i\frac{d}{dt}|\Psi_I(t)\rangle = -\hat{H}_0 e^{i\hat{H}_0 t}|\Psi_S\rangle + e^{i\hat{H}_0 t} \underbrace{i\frac{d}{dt}|\Psi_S\rangle}_{= \hat{H}_0 + \hat{H}_{int}}$$

$$= \left(-\cancel{\hat{H}_0 e^{i\hat{H}_0 t}} + e^{i\hat{H}_0 t}(\cancel{\hat{H}_0} + \hat{H}_{int}) e^{-i\hat{H}_0 t} e^{i\hat{H}_0 t}\right)|\Psi_S$$

$$= \left(\hat{H}_{int}\right)_I |\Psi_s\rangle$$

Propagator $\hat{U}(t,t_0)\,|\Psi_I(t_0)\rangle = |\Psi_I(t)\rangle$

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_{int}\,\hat{U}(t,t_0)$$

$$\hookrightarrow \hat{U}(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_{int}(t')\,dt'\right)$$

$$T\{O_1(t_1)\,O_2(t_2)\} = \begin{cases} O_1(t_1)\,O_2(t_2), & t_1 > t_2 \\ O_2(t_2)\,O_1(t_1), & t_2 > t_1 \end{cases}$$

S-matrix in- and out-states

$$\begin{bmatrix} |i\rangle & : t = -\infty \\ |f\rangle & : t = +\infty \end{bmatrix}$$

Eigenstates of free $\hat{H}_0$

Transition amplitude $|i\rangle \longrightarrow |f\rangle$

S-matrix $\lim\limits_{t_\pm \to \pm\infty} \langle f(t_+)|\,\hat{U}(t_+,t_-)\,|\,i(t_-)\rangle$

What is $\hat{H}_{int}$? $\sim \phi(x_1)\,\phi(x_2)\ldots\phi(x_n)$

$$|i\rangle = \prod_{i=1}^{n}\left(\sqrt{2\omega_{p_i}}\;\hat{a}^{+}_{p_i}\right)|0\rangle$$

$|f\rangle$ similar

$H \sim a^{+}a$ normal ordering $:\ldots:$

$$T\{\varphi(x_1)\,\varphi(x_2)\dots\varphi(x_n)\}$$

$$= :\varphi(x_1)\dots\varphi(x_n): + \sum \text{ all contractions}$$

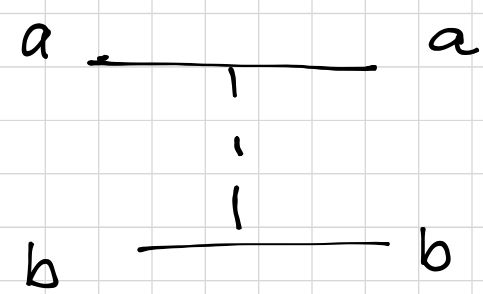
$$\underbrace{\varphi(x_e)\dots\varphi(x_m)}_{\text{Feynman prop.}}$$

Wick's theorem

Feynman rules $\longrightarrow$ Feynman diags.

F.P. — central concept in QFT

Interaction — simple picture, eg.



contraction $\rightarrow$ 1 p. exch.
mass of exchanged particle $\rightarrow$ range of int.

Unification of short and long-range forces.

Central to interaction picture: asymptotic states are eigenstates of the free Hamiltonian ($\hat{H}_0$). Is this still useful? We know that in QFT we cannot work with fixed # of particles. This question is answered by the LSZ reduction formula