Introduction to Theoretical Particle Physics: WS 2022/2023: Exercise sheet 8

20.01.2023

Exercise 1: Higgs mechanism in non-abelian models (100+25 points)

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(25 points) Consider $SU(2)$ theory interacting with a scalar field with the "Mexican hat" potential:

$$\mathcal{L} = -\frac{1}{4} F_{a;\mu\nu} F_a^{\mu\nu} + (D^\mu \phi)^+ (D_\mu \phi) - \lambda \left(\phi^+ \phi - v^2\right)^2$$

v is the real constant. The covariant derivative is:

$$D_\mu \phi = \partial_\mu \phi + ie A_{a;\mu} \frac{\sigma_a}{2} \phi$$

Prove using the gauge invariance that the excitations around the minima can be written in the form:

$$\phi \approx \begin{pmatrix} 0 \\ v + \varphi \end{pmatrix}$$

Where φ is the real function.

(b)(25 points) Expand the Lagrangian to obtain the following low-energy expression:

$$\mathcal{L} \approx -\frac{1}{4} F_{a;\mu\nu} F_a^{\mu\nu} + \frac{1}{4} e^2 v^2 A_a^\mu A_{a;\mu} + (\partial^\mu \varphi)(\partial_\mu \varphi) - 4\lambda v^2 \varphi^2 + \text{higher order terms}$$

I.e. all three generators are broken in this case.

(c)(25 points) Now consider a theory with $SU(2) \otimes U(1)$ invariance:

$$\mathcal{L} = -\frac{1}{4} F_{U(1)}^2 - \frac{1}{4} F_{SU(2)}^2 + (D^\mu \phi)^+ (D_\mu \phi) - \lambda \left(\phi^+ \phi - v^2\right)^2$$

The covariant derivative is:

$$D_\mu \phi = \partial_\mu \phi + ie_1 A_\mu \frac{\phi}{2} + ie_2 A_{a;\mu} \frac{\sigma_a}{2} \phi$$

The factor 1/2 for $U(1)$ field was added to make the expression more symmetric. Prove that it can be written in the following form:

$$D_\mu \phi = \partial_\mu \phi + ie \left(A_\mu - \operatorname{tg} \theta_W Z_\mu \right) t_A \phi + i \left(\frac{e_2}{\cos \theta_W} Z_\mu \frac{\sigma_3}{2} + e_2 W_{+;\mu} \frac{t_+}{2} + e_2 W_{-;\mu} \frac{t_-}{2} \right) \phi$$

Where denoted:

$$\begin{aligned}
e &= e_1 \cos \theta_W = e_2 \sin \theta_W \\
A &= \cos \theta_W A_{U(1)} + \sin \theta_W A_{SU(2);\,3} \\
Z &= - \sin \theta_W A_{U(1)} + \cos \theta_W A_{SU(2);\,3} \\
W_\pm &= A_{SU(2);\,1} \mp i A_{SU(2);\,2}
\end{aligned}$$

And additionally:

$$t_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Hint: use the relation:

$$\frac{1}{2} = t_A - \frac{\sigma_3}{2}$$

(d)(25 points) Repeat the part b) for $SU(2) \times U(1)$ model given above. Prove that the answer in terms of fields $W_\pm, Z, A$ is:

$$\begin{aligned}
&\mathcal{L} \approx -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 - \frac{1}{4} \left(\partial_\mu Z_\nu - \partial_\nu Z_\mu \right)^2 - \\
&- \frac{1}{2} \partial_\mu W_{+;\nu} \cdot \partial^\mu W_-^\nu + \frac{1}{2} \partial^\mu W_{+;\mu} \cdot \partial_\nu W_-^\nu + \left(\partial_\mu \varphi \right)^2 + \\
&+ \frac{1}{4 \cos^2 \theta_W} e_2^2 v^2 Z_\mu^2 + \frac{1}{4} e_2^2 v^2 W_{+;\mu} W_-^\mu - 4 \lambda v^2 \varphi^2 + \text{higher order terms}
\end{aligned}$$

(e*)(Bonus - 25 points) Write the action for spin-0 and spin-1/2 fields in a curved space-time in the presence of $SU(N)$ symmetry.

Literature

1. No literature this time.