Introduction to Theoretical Particle Physics: WS 2022/2023: Exercise sheet 3

25.11.2022

Exercise 1: Non-abelian gauge theories (100+25 points)

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(45 points) Consider a set of matter fields $\phi_a(x)$ (either scalar or spinor with respect to the Lorentz group) under a global (i.e. without coordinates dependence) unitary transformation:

$$\phi_a(x) \to \omega_{ab}\phi_b(x); \quad \omega^*_{ba} = (\omega_{ab})^{-1}$$

Which can be considered as a vector rotation in a so-called color (or internal) space, denoted by the index a.
In other words, the matter field $\phi_a(x)$ is defined in a combined Lorentz $\otimes$ color space. The derivative obviously transforms in the following way:

$$\partial^\mu \phi_a(x) \to \omega_{ab}\partial^\mu \phi_b(x)$$

So the Lagrangian (either scalar or spinor) remains invariant.
Further we will mostly use the componentless form, keeping in mind that ϕ is now a vector object with respect to the color space:

$$\phi(x) \to \omega\phi(x); \quad \partial^\mu\phi(x) \to \omega\partial^\mu\phi(x); \quad \omega^+ = \omega^{-1}$$

Next, we are interested in the local transformations with the coordinate dependence of matrix ω, i.e. $\omega = \omega(x)$. To keep the Lagrangian invariant, we introduce a covariant derivative and a gauge field A^μ (which is necessarily a vector field with respect to the Lorentz group):

$$D^\mu\phi(x) = \partial^\mu\phi(x) + A^\mu(x)\phi(x)$$

Prove that the Lagrangian remains invariant if the gauge field transforms as follows:

$$A^\mu(x) \to \omega A^\mu(x)\omega^+ + \omega\partial^\mu\omega^+$$

Because in this case we have (compare with the case of global transformation):

$$D^\mu\phi(x) \to \omega D^\mu\phi(x)$$

(b)(25 points) Prove that the field strength tensor constructed in the following way:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + [A^{\mu}, A^{\nu}]$$

Transforms as follows:

$$F^{\mu\nu} \to \omega F^{\mu\nu} \omega^{+}$$

(c)(30 points) Note that field $A^{\mu}(x)$ is a matrix object with respect to the color space. It can be decomposed in a basis of matrix t_a, which are called generators:

$$A^{\mu}(x) = ieA_a^{\mu}(x)\, t_a$$

Upper and lower indices are indistinguishable (i.e. Euclidean). Generators are Hermitian matrices normalized by the condition:

$$\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab}$$

And additionally we assume that:

$$[t_b, t_c] = i f_{bca} t_a$$

Where f_{ijk} are called the structure constants.
Prove that for the field strength tensor $F^{\mu\nu} = ieF_a^{\mu\nu} t_a$ we have:

$$F_a^{\mu\nu} = \partial^{\mu} A_a^{\nu} - \partial^{\nu} A_a^{\mu} - e f_{bca} A_b^{\mu} A_c^{\nu}$$

And that gauge field Langrangian:

$$\mathcal{L} = \frac{1}{2e^2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F_{\mu\nu;a}F_a^{\mu\nu}$$

Is invariant.
Note: unitary transformation $\omega(x)$ can be written in the following form:

$$\omega(x) = e^{i\theta(x)}$$

Where $\theta(x)$ is a Hermitian matrix. Of course it can also be decomposed in the basis:

$$\theta(x) = ie\theta_a(x)\, t_a$$

(d*)(Bonus - 25 points) Prove Furry's theorem, which states that an arbitrary diagram with an odd number of gauge field is equal to zero, i.e:

$$\langle\Omega| T\left[\hat{A}(x_1)\ldots\hat{A}(x_{2n+1})\right]|\Omega\rangle = 0$$

Hint: think about the charge conjugation.

Appendix A Note on the notation

Another common form of notation is:

$$D^{\mu}\phi\left(x\right)=\partial^{\mu}\phi\left(x\right)+ieA^{\mu}\left(x\right)\phi\left(x\right)$$
$$A^{\mu}=A_{a}^{\mu}t_{a}$$
$$F^{\mu\nu}=F_{a}^{\mu\nu}t_{a}$$
$$\mathcal{L}=-\frac{1}{2}\operatorname{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)=-\frac{1}{4}F_{\mu\nu;a}F_{a}^{\mu\nu}$$

In some cases charge can be taken with an opposite sign. It also can be denoted as g (Schwartz M.D.; Ryder L.H.).

Literature

1. Quantum Field Theory and the Standard Model, Schwartz M.D. - chapter 25.
2. Quantum field theory, Ryder L.H. - chapter 3.