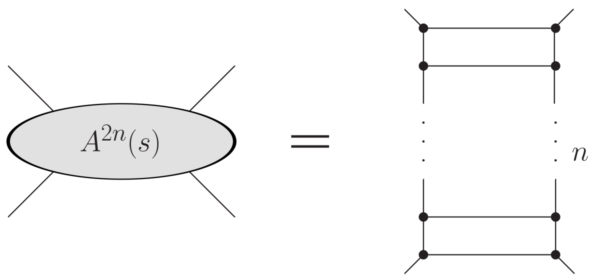
Exercise 6
total of 200 points – Christmas bonus

December 21, 2022

1 Ladder resummation in the scalar ϕ^3 theory

(a) **50 points**:
Compute the ladder diagram of rank n (in Figure below) in the scalar ϕ^3 theory with mass m and coupling g in the forward kinematics and at high $s \gg m^2$.

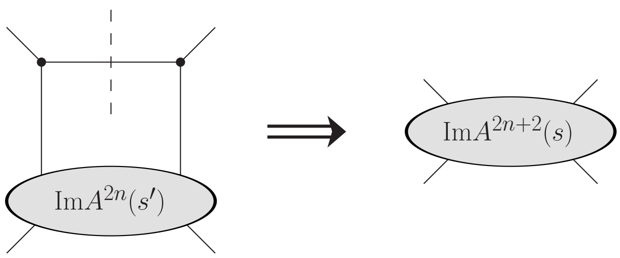


Prove that the result reads,

$$\frac{1}{\pi}\text{Im}\, A^{(2n)}(s \to \infty) = \frac{g^2}{s}\left(\frac{g}{4\pi m}\right)^2 \frac{1}{(n-2)!}\left[\left(\frac{g}{4\pi m}\right)^2 \log\frac{s}{m^2}\right]^{n-2} \theta(s-(n\,m)^2), \quad n \geq 2$$

$$\text{Re}\, A^{(2n)}(s \to \infty) = -\frac{g^2}{s}\frac{1}{(n-1)!}\left[\left(\frac{g}{4\pi m}\right)^2 \log\frac{s}{m^2}\right]^{n-1},$$

Tip: use the method of mathematical induction by computing the imaginary part of the 1-loop diagram by applying the cutting rules first, and extend this result to the next order as shown below.



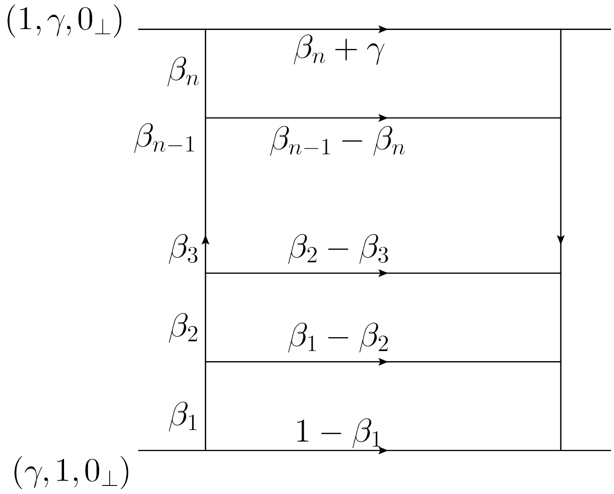
Then reconstruct the real part from a dispersion relation of the form

$$\text{Re}\; A^{(2n)}(s) = \frac{1}{\pi}\mathcal{P}\int_{(nm)^2}^{\infty} \frac{ds'}{s'-s}\text{Im}\; A^{(2n)}(s'), \tag{1}$$

using the master integral $\mathcal{P}\int_0^1 \frac{dx}{x-y} \ln^n \frac{1}{x} = -n!\ Li_{n+1}(1/y)$, with the polylogarithm function asymptotics $Li_n(z \to \infty) = -1/n! \ln^n(z)$

(b): **50 points**

Obtain the same result by method of regions by locating poles in the complex plane, doing the Wick rotation and computing the residue. Work in the light-cone coordinates $k^\mu = (\alpha n^\mu + \beta \bar{n}^\mu + k_\perp^\mu) \equiv (\alpha, \beta, \vec{k}_\perp)$ with the lightlike Sudakov vectors $n = \sqrt{s}/2(1,0,0,1)$ and $\bar{n} = \sqrt{s}/2(1,0,0,-1)$, which obey $n^2 = \bar{n}^2 = 0$, $2n\bar{n} = s$, thus the scalar product of two 4-vectors is $(a_1 a_2) = s/2(\alpha_1\beta_2 + \alpha_2\beta_1) - \vec{a}_\perp \vec{b}_\perp$. Show that the asymptotic result is obtained with the strong ordering in one of the lightlike coordinates, e.g. β, such that $1 \gg \beta_1 \gg \beta_2 \gg \ldots \gg \beta_n \gg \gamma$, with $\gamma = m^2/s$.



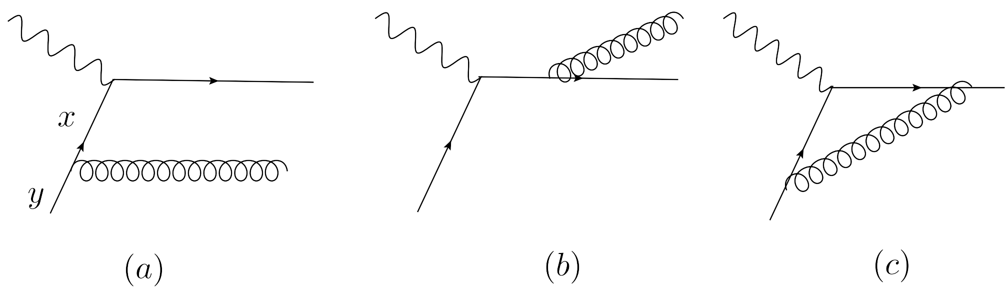
2 Operator Product Expansion for DIS and DGLAP evolution of the valence quark distributions

(a) **30 points**:

Follow the derivation sketched in Lecture 10: express the forward quark Compton amplitude

$$t^{\mu\nu} = i \int d^4x e^{iqx} \langle p_q | T\{j^\mu(x) j^\nu(x)\} | p_q \rangle \tag{2}$$

in terms of the basis of twist-2 spin-n operators $\hat{O}_n^{\mu_1\mu_2\cdots\mu_n} = \psi(x)\gamma^{\{\mu_1} i\partial^{\mu_2} \ldots i\partial^{\mu_n\}}\psi(x)$, where $\{\ldots\}$ in the superscript denotes the symmetrization over the Lorentzs indices and removal of the traces. Evaluate the matrix elements of these operators between the prooton states to obtain the expression for the proton DIS structure functions at leading twist and leading order in the strong coupling constant (parton model result).



(b) **70 points**:

(Following Lecture 11) – Compute the leading-logarithm evolution of the quark distributions functions at NLO in the strong coupling constant by isolating the collinear log singularity $\sim \ln(Q^2/\mu^2)$ where μ is the effective quark mass kept finite to regularize the singularity. Obtain the IR divergent part of diagram (c) and observe that its contribution to the squared amplitude (fill in details in Chapter 32.2 in Schwartz's book) cancels that of the interference of the diagrams (a,b).

Merry Christmas & Happy New Year!