Exercise sheet 4
Theoretical Physics 3: WS2022/2023
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Exercise 1. Fourier transform (45 points)

We define the (spatial) Fourier transform of a wave function $\Psi(x,t)$, and its corresponding inverse transform as:

$$\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ikx}\, dx$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k,t) e^{ikx}\, dk$$

Using this definition, calculate the Fourier transforms of the following functions:

a) *(2 p.)* $\Psi(x) = \delta(x)$ and $\Psi(x) = \delta(x - x_0)$

b) *(2 p.)* $\Psi(x) = a = \text{const}$

c) *(4 p.)* $\Psi(x) = \cos(x)$

d) *(7 p.)* $\Psi(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

e) *(10 p.)* Apply the Fourier transform to write down the Schrödinger equation representation for the quantum harmonic oscillator in the k-domain.

Hint: use the identity:

$$\int_{-\infty}^{\infty} x e^{ax}\, dx = \frac{\partial}{\partial a} \int_{-\infty}^{\infty} e^{ax}\, dx$$

f) *(10 p.)* Write down the Schrödinger equation with an arbitrary potential in the k-domain. Assume the potential is expandable in the power series $V(x) = \sum_n a_n x^n$.

g) *(10 p.)* Solve the one-dimensional free particle Schrödinger equation in the k-domain. Then apply a reverse Fourier transform to write down the general solution in the spatial domain.

Exercise 2. Double δ-potential (55 points)

Consider the following one-dimensional model potential for a molecule with one doubly degenerate state:

$$V(x) = -V_0 a \left(\delta(x-a) + \delta(x+a)\right),$$

Where V_0 and a are real parameters.

a) *(5 p.)* Apply the Fourier transform to the corresponding Schrödinger equation, $\hat{H}(x)\psi(x) = E\psi(x)$. Show that in the momentum space it becomes:

$$\frac{\hbar^2 k^2}{2m}\phi(k) - \frac{V_0 a}{\sqrt{2\pi}}\left(\psi(a)e^{-ika} + \psi(-a)e^{ika}\right) = E\phi(k),$$

Where:

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ikx}\psi(x) \quad \text{and} \quad \psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx}\phi(k)$$

b) *(15 p.)* Using the obtained Schrödinger equation in the momentum space, find the bound states of the system in the coordinate space. How many bound states does the system have?
Hint: The solution must be consistent at the two points $x = \pm a$.

c) *(10 p.)* For $V_0 a = \frac{\hbar^2}{ma}$, find the energies of the stationary states and sketch the corresponding wave functions.
Hint: Use the fact that there are odd and even solutions.

d) *(10 p.)* Discuss the role of the parameter a on the stationary states (consider $a \to 0$ and $a \to \infty$).

e) *(15 p.)* Find the reflection and transmission coefficients for a beam of particles on this potential.