Exercise sheet 3
Theoretical Physics 3: QM WS2022/2023
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Exercise 1. Harmonic oscillator: ladder operators (35 points)

Consider a quantum harmonic oscillator, the time-independent ground state wave function of which is given by:

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega x^2}{2\hbar}} \equiv \alpha e^{-\frac{y^2}{2}}$$

Where, for further simplicity, we have introduced $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$ and dimensionless variable $y = \sqrt{\frac{m\omega}{\hbar}}x$.

a) *(5 p.)* Using explicit definition of the raising ladder operator:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega m}}(-i\hat{p} + m\omega\hat{x}) \equiv \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}y} + y\right)$$

Derive expression for the first excited state wave function ψ_1 and check its orthogonality to ψ_0.

b) *(20 p.)* Compute $\langle x\rangle, \langle p\rangle, \langle x^2\rangle$ and $\langle p^2\rangle$, for the states ψ_0 and ψ_1 by explicit integration.

c) *(5 p.)* Check the uncertainty principle for these two states.

d) *(5 p.)* Compute expectation values of the kinetic energy $\langle T\rangle$ and the potential energy $\langle V\rangle$. Check these to sum up to $\langle H\rangle$.

Exercise 2. Harmonic oscillator: power series method (65 points)

The quantum harmonic oscillator problem can be solved using the power series method. One starts with the stationary Schrödinger equation ($\psi'' \equiv \frac{d^2\psi}{dx^2}$):

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\,\psi(x)$$

a) *(10 p.)* To simplify the initial problem, rewrite the equation using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}}x, \quad \varepsilon = E/\hbar\omega$$

Further on, define $\varphi(y) = c\psi(x)$ and find c, such that $\varphi(y)$ is normalized.

b) *(10 p.)* Investigate the asymptotical behaviour of the equation for large y. Show that for $y \to \infty$:

$$\varphi(y) \sim e^{-\frac{y^2}{2}}$$

c) *(10 p.)* We can now explicitly isolate the asymptotic behaviour of the unknown function:

$$\varphi(y) = h(y)\, e^{-\frac{y^2}{2}}$$

Derive the following equation on $h(y)$:

$$h'' - 2yh' + (2\varepsilon - 1)h = 0$$

d) *(15 p.)* At this point, assume that $h(y)$ can be written as an infinite power series in y:

$$h(y) = \sum_{m=0}^{\infty} a_m\, y^m$$

Derive the recurrence relation between the coefficients a_m and show that there are two sets of independent solutions (*even* and *odd*).

e) *(15 p.)* Prove that, in order for the wave function to be finite and normalizable, one has to imply that the infinite series must be "cut off" at some finite integer n: $a_{m>n} = 0$.
(*Hint*: consider Maclaurin expansion of e^{y^2} and compare it to the series behaviour for large y).

f) *(5 p.)* Using the previous conclusion, show that the energy is quantized $E_n = (n + \frac{1}{2})\hbar\omega$.

The obtained polynomials $h_n(y)$ are proportional to *Hermite polynomials* $H_n(y)$. The orthonormal set of solutions of the initial stationary Schrödinger equation then reads:

$$\psi_n(x) = \left(2^n n! \sqrt{\frac{\pi\hbar}{m\omega}}\right)^{-\frac{1}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}, \quad n = 0, 1, 2, ...$$

(Bonus) Exercise 3. Supersymmetric QM (50 points)

We consider in this exercise a generalization of the raising and lowering operator method. For a given potential $V_-(x)$, the idea is to construct a partner (also called supersymmetric) potential $V_+(x)$ which has the same energy eigenvalues, except for the ground state. Without loss of generality, we can shift the potential $V_-(x)$ so that the corresponding ground state $\psi_0(x)$ has zero energy $E_0^- = 0$.

a) *(10 p.)* Show that the Hamiltonian $\hat{H}_- = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-$ can be written in the form:

$$\hat{H}_- = \hat{A}^+ \hat{A}$$

With:

$$\hat{A}^+ \equiv \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)$$
$$\hat{A} \equiv \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)$$

And $\psi_0' \equiv \frac{d}{dx}\psi_0$.

b) *(10 p.)* Consider the partner Hamiltonian operator $\hat{H}_+ = \hat{A}\hat{A}^+$ which can also be defined as $\hat{H}_+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+$. Show that the partner potential V_+ is related to V_- as follows:

$$V_+ = V_- - \frac{\hbar^2}{m}\frac{d}{dx}\left(\frac{\psi_0'}{\psi_0}\right)$$

c) *(15 p.)* Show that $\hat{H}_-$ and $\hat{H}_+$ have the same spectrum, except for the ground state (*Hint*: Consider the states $\hat{A}\psi_n^-$ and $\hat{A}^+\psi_n^+$ with $\psi_n^\pm$ being eigenstates of $\hat{H}_\pm$). Write the eigenstates ψ_n^+ and energies E_n^+ in terms of ψ_n^- and E_n^-.

d) *(15 p.)* Consider a particle in the infinite square potential well:

$$V_-(x) = \begin{cases} V_0 & \text{for} \quad 0 \leq x \leq a \\ +\infty & \text{otherwise,} \end{cases}$$

Find V_0 such that the ground state has zero energy. Derive the partner potential $V_+(x)$. Write down the properly normalized eigenstates $\psi_n^+(x)$. Explain why the existence of partner potentials may be useful.