# Introduction to Theoretical Particle Physics: WS 2022/2023: Exercise sheet 1

11.11.2022

## Exercise 1: Introduction to path integral methods (100+25 points)

**(0)(0 points)** How much time did you spend in solving this exercise sheet?

**(a)(50 points)** Consider the path integral in quantum mechanics:

$$K\left(x_b, t_b; x_a, t_a\right) = \int_{x[t_a]=x_a}^{x[t_b]=x_b} e^{iS[x]}\, Dx(t); \quad S[x] = \int_{t_a}^{t_b} \left(\frac{m\dot{x}^2}{2} - V\right) dt$$

The steepest descent method (also known as semiclassical approximation) states that the main contribution to this integral comes from the path where $\delta S = 0$, i.e. from the classical trajectory $x_{cl}$.

Expand the exponent around this classical path to prove that:

$$K\left(x_b, t_b; x_a, t_a\right) \approx \frac{const}{\sqrt{\mathrm{Det}\left(-m\frac{\partial^2}{\partial t^2} - \frac{\partial^2 V}{\partial x^2}\right)}} e^{iS_{cl}}$$

Where $S_{cl}$ is the classical action for the evolution from the point $[x_a, t_a]$ to the point $[x_b, t_b]$, the constant refers to the normalization of the path integral.

*Note*: this method can also be easily generalized to be applied to QFT problems.

**(b)(50 points)** Consider the harmonic oscillator:

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right)$$

First prove that the classical action for the evolution from the point $[x_a, t_a]$ to the point $[x_b, t_b]$ is:

$$S_{cl} = \frac{m\omega}{2\sin\left(\omega T\right)}\left[\left(x_b^2 + x_a^2\right)\cos\left(\omega\left(t_b - t_a\right)\right) - 2x_b x_a\right]$$

After that calculate the path integral using the results of the previous section and prove that the semiclassical approximation in this case gives an exact answer.

**(c*)(Bonus - 25 points)** Use Wick rotation to obtain the energy spectrum of the harmonic oscillator:

$$E_n = \omega \left( n + \frac{1}{2} \right)$$

*Hint*: use the partition function.

*Tip*: scalar field can be considered as an infinite set of harmonic oscillators. Using the results of this exercise sheet, you can easily calculate the path integral for such system and get the energy spectrum.

## Literature

1. Quantum Mechanics and Path Integrals, R. P. Feynman and A. R. Hibbs - chapters 2 and 3 (also check the problems list).