Practice exam
Theoretical Physics 6a (QFT): SS 2022

28.07.2022

Exercise 1. (20 points): Dilation current for spinors

Real-valued Lagrangian for spinor field is:

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi$$

(a)(10 points) Prove that in case of $m = 0$ the action $\mathcal{S} = \int \mathcal{L}\, d^4x$ has a dilation symmetry, i.e. it is invariant under the transformation:

$$x^\mu \to x^\mu e^\lambda$$
$$\psi \to \psi e^{-\frac{3}{2}\lambda}$$

(b)(10 points) Derive the corresponding Noether current:

$$j^\mu = x_\nu T^{\mu\nu}$$

And prove that the conservation of this current implies the energy-momentum tensor $T^{\mu\nu}$ to be traceless.

Exercise 2. (15 points): Spontaneous breaking of discrete symmetry

Consider the real scalar theory:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right) - V$$
$$V = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2$$

It has Z_2 symmetry, i.e. it is symmetric under the transformation $\phi \to -\phi$.

(a)(5 points) Prove that this potential has a wrong mass sign compared to the usual scalar theory and sketch the potential as a function of ϕ.

(b)(10 points) Prove that for small deviations from the equilibrium position:

$$\phi(x) = v + \chi(x)$$

The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \chi\right)\left(\partial_\mu \chi\right) - V'$$
$$V' = \lambda v^2 \chi^2 + \lambda v \chi^3 + \frac{\lambda}{4}\chi^4$$

Which has no Z_2 symmetry anymore. The sign of the mass term of the field χ is now correct.

Exercise 3. (30+10 points): Møller scattering in scalar QED

The Lagrangian for scalar QED with complex scalar field ϕ, describing the interaction of charged scalars with the photon field A^μ, is given by:

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Where $D^\mu \phi$ denotes the covariant derivative:

$$D^\mu \phi = (\partial^\mu + ieA^\mu)\phi$$

(a)(5 points) Determine the interaction terms and deduce the Feynman rules for this theory.

(b)(5 points) Derive the tree-level matrix element for the process $\phi^-\phi^- \to \phi^-\phi^-$ and draw the two contributing Feynman diagrams.

(c)(10 points) Show that the squared matrix element for this process is given by:

$$|\mathcal{M}|^2 = e^4 \left[\frac{s-u}{t} + \frac{s-t}{u}\right]^2$$

Where s, t, u are the Mandelstam invariants for the process $p_1 + p_2 \to p_3 + p_4$:

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - p_3)^2\,, \quad u = (p_1 - p_4)^2$$

(d)(10 points) Show that the differential cross section in the center-of-momentum frame can be written as:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left[\frac{s-u}{t} + \frac{s-t}{u}\right]^2$$

(Bonus)(10 points) Using crossing symmetry, write the squared matrix element and the cross section for scalar Bhabha process, i.e. $\phi^-\phi^+ \to \phi^-\phi^+$.

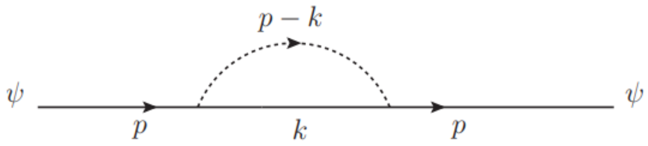
Exercise 4. (35+10 points): 1-loop correction to the propagator in pseudoscalar Yukawa theory

Consider the interaction between a scalar field ϕ (with mass M) and a spin 1/2 field ψ (with mass m) described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4$$

(a)(5 points) Determine the interaction terms and deduce the Feynman rules for this theory.

(b)(10 points) The only non-vanishing 1-loop correction to the fermion propagator is presented by the following diagram:



Arrows represent the direction of momenta.
Calculate the corresponding self-energy, using the momenta labels as indicated on the figure.

(c)(10 points) Use the Feynman parametrization and perform the one-loop integral using dimensional regularization. Show that the result can be expressed as:

$$\mathcal{M} = g^2 \mu^{4-D} \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)} \left[\int_0^1 dx \frac{(x\not{p} - m)}{\Delta^{2-D/2}} \right]$$

And $\Delta(x)$ is given by:

$$\Delta(x) = -p^2 x\,(1-x) + m^2(1-x) + M^2 x.$$

(d)(10 points) Define the counter-terms in dimensional regularization in the following way:

$$\mathcal{L}_{CT} = \frac{(Z_1-1)}{2}(\partial_\mu\phi)(\partial^\mu\phi) - (Z_2-1)\,\frac{1}{2}M^2\phi^2 + (Z_3-1)\,i\bar{\psi}\not{\partial}\psi - (Z_4-1)\,m\bar{\psi}\psi -$$
$$-\,(Z_5-1)\,ig\mu^{\frac{4-D}{2}}\bar{\psi}\gamma^5\psi\phi - (Z_6-1)\,\frac{\lambda\mu^{4-D}}{4!}\phi^4$$

Find the counter-terms in MS scheme for the self-energy diagram above.

(Bonus)(10 points) Perform the Feynman integral in the limit $m^2 \gg M^2$ for on-shell particle ($p^2 = m^2$) and calculate the finite correction to the renormalized self-energy in this limit. Make use of the integral:

$$\int_0^1 dx\; x^n \ln\left[(1-x)^2\right] = \frac{-2}{1+n} \sum_{j=1}^{n+1} \frac{1}{j}$$

Useful Formulas

The formula for the cross section for the process $a + b \to c + d$ is given by

$$d\sigma = \frac{1}{(2E_a)(2E_b)v_{rel}} \frac{d^3\vec{p}_c}{(2\pi)^3(2E_c)} \frac{d^3\vec{p}_d}{(2\pi)^3(2E_d)} (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)|\mathcal{M}|^2,$$

where the four-momenta for each particle are given by $p_i = (E_i, \vec{p}_i)$, where v_{rel} stands for the incident flux, and $\mathcal{M}$ is the invariant amplitude. A useful identity for the δ-distribution is

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|} \qquad \text{with } f(x_0) = 0.$$

Formulas used in one-loop calculations are

$$\frac{1}{D_1 D_2} = \int_0^1 dx \frac{1}{[(1-x)D_1 + xD_2]^2},$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - \Delta + i\epsilon)^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \frac{(-1)^n}{\Delta^{n-D/2}},$$

$$\Gamma(z+1) = z\Gamma(z),$$

$$\Gamma(-1+\varepsilon) = -1 \left\{ \frac{1}{\varepsilon} + 1 - \gamma_E + \mathcal{O}(\varepsilon) \right\},$$

with Euler constant $\gamma_E \approx 0.577$.