Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 12

11.07.2022

Exercise 1. (100+25 points): Running couplings

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(100 points) Consider the massive real scalar theory in the dimensional regularization:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\mu^{4-d}\lambda}{4!}\phi^4$$

The factor μ has a dimension of mass, so that the coupling constant remains dimensionless.
It is advised to write the counterterms in the following form:

$$\Delta\mathcal{L} = \frac{Z_2 - 1}{2}(\partial\phi)^2 - (Z-1)\, m^2\phi^2 - (Z_4 - 1)\frac{\mu^{4-d}\lambda}{4!}\phi^4$$

Which gives the bare Lagrangian:

$$\mathcal{L} + \Delta\mathcal{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 = \frac{Z_2}{2}(\partial\phi)^2 - \frac{Z}{2}m^2\phi^2 - Z_4\frac{\mu^{4-d}\lambda}{4!}\phi^4$$

With bare parameters:

$$\phi_0 = \sqrt{Z_2}\phi$$
$$m_0^2 = \frac{Z}{Z_2}m^2$$
$$\lambda_0 = \mu^{4-d}\frac{Z_4}{Z_2^2}\lambda$$

Calculate all one-loop diagrams and prove that in MS scheme counterterms are the following ($D = 4 - 2\varepsilon$):

$$Z_4 = 1 + \frac{\lambda}{16\pi^2}\frac{3}{2\varepsilon}$$
$$Z = 1 + \frac{\lambda}{16\pi^2}\frac{1}{2\varepsilon}$$
$$Z_2 = 1$$

Also prove that the running coupling and mass:

$$\lambda = \frac{1}{\beta_0 \ln\left(\frac{\mu}{\Lambda}\right)}$$
$$m^2(\mu) = m^2(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m}$$

Have the following coefficients:

$$\beta_0 = -3\left(\frac{1}{4\pi}\right)^2$$
$$\gamma_m = \frac{\lambda}{16\pi^2}$$

Hint: there are 4 diagrams, but 3 of them have the same structure.

(b*)(Advanced level problem for those who are interested - 25 points) List all renormalizable interactions with scalars, spinors and massless vector fields in $D = 1+3$. Why massive vector field is non-renormalizable? Also prove that in $D = 1+1$ an arbitrary polynom of scalar fields and massless vector fields appears to be renormalizable, but spinor field is still restricted with the mass term only.
Note: spin 3/2 field leads to solutions which propagate with faster than light velocities. Gravity is also non-renormalizable because $[G_{Newton}] = -2$. And so on - only a few options are actually available.

Literature

1. See exercise classes and lecture notes.
2. Quantum Field Theory, Lewis Ryder (chapter 9, but be careful - there are a couple of typos in this section).
3. Quantum Field Theory and the Standard Model (Schwartz M.D.), chapters 21-23.