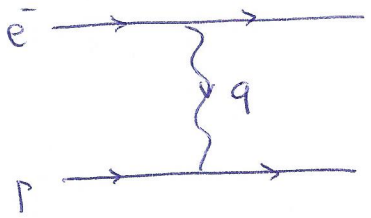
UEHLING POTENTIAL & LAMB SHIFT IN H-ATOMS

$\Rightarrow$ **COULOMB POTENTIAL** e.g. IN H-ATOM



$(ie\gamma^\mu)$

$-\frac{ig_{\mu\nu}}{q^2+i\varepsilon}$

$(ie\gamma^\nu)$

POTENTIAL $\tilde{V}_c(q^2) = \frac{e^2}{q^2}$

STATIC LIMIT $q^0 \ll |\vec{q}|$ (IN ATOM)

$q^2 \simeq -\vec{q}^2$

$\tilde{V}_c(-\vec{q}^2) = -\frac{e^2}{\vec{q}^2}$

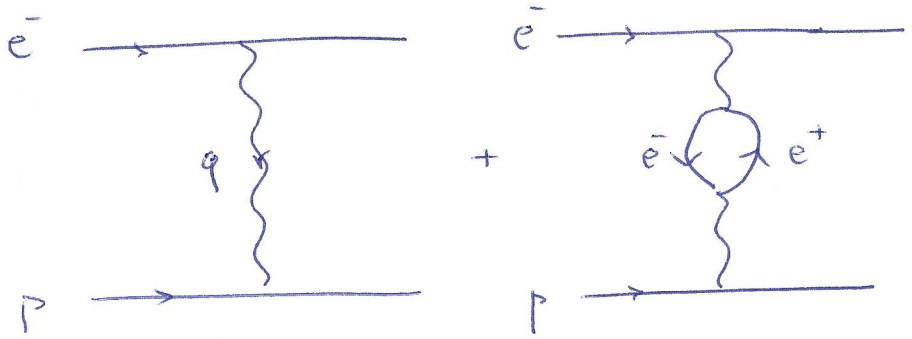
COORDINATE SPACE

$$V_c(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \tilde{V}_c(-\vec{q}^2)$$

$$= -\frac{e^2}{4\pi r} = -\frac{\alpha}{r}$$

WITH $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$

$\Rightarrow$ **UEHLING POTENTIAL.** ORDER e^2 CORRECTION TO COULOMB POTENTIAL IN H-ATOM



$(ie\gamma^\mu)$

$\frac{-ig_{\mu\nu}}{q^2[1-\Pi(q^2)]}$

$(-ie\gamma^\nu)$

POTENTIAL $\tilde{V}(q^2) = \frac{e^2}{q^2[1-\Pi(q^2)]}$

$\Pi(q^2)$ IS LOGARITHMICALLY DIVERGENT

$\Pi(q^2) = \Pi(0) + \underbrace{\Pi(q^2) - \Pi(0)}_{\text{FINITE } O(e^2)}$

$\Pi(0)$ ↑ CONTAINS DIVERGENCE AS POLE $\frac{1}{\varepsilon}$

$\varepsilon = 2 - \frac{D}{2}$

$\tilde{V}(q^2) = \boxed{\frac{e^2}{[1-\Pi(0)]}} \cdot \frac{1}{q^2} \frac{1}{\left[1 - \frac{\Pi(q^2)-\Pi(0)}{1-\Pi(0)}\right]}$

e_R^2 ↑ RENORMALIZED (PHYSICAL) CHARGE

CORRECTION TO ORDER $O(e_R^2)$

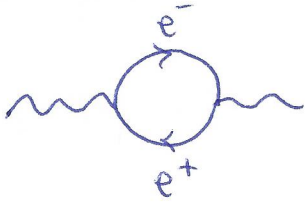
$$\tilde{V}(q^2) = \frac{e_R^2}{q^2}\left[1 + \underbrace{\frac{\Pi(q^2)-\Pi(0)}{1-\Pi(0)}}_{\text{UEHLING POTENTIAL } \Pi_R(q^2)} + O(e_R^4)\right]$$

$$= \tilde{V}_C + \tilde{V}_U + O(e_R^6)$$

$\tilde{V}_C$: $O(e_R^2)$; $\tilde{V}_U$: $O(e_R^4)$

- $$\frac{\Pi(q^2)-\Pi(0)}{1-\Pi(0)} = \frac{e_R^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1-\frac{q^2}{m^2}x(1-x)\right]$$

1 LOOP EXPRESSION



$$Q^2 = -q^2 \simeq \bar{q}^2$$

↑ IN STATIC LIMIT $(q^0 \ll |\bar{q}|)$

TO WORK OUT x- INTEGRAL

DEFINE $\tau \equiv \frac{Q^2}{4m^2}$

$$\ln\left[1-\frac{q^2}{m^2}x(1-x)\right]$$

$$= \ln\left(\frac{Q^2}{m^2}\left[\frac{m^2}{Q^2} + x(1-x)\right]\right)$$

$$= \ln\frac{Q^2}{m^2} + \ln\underbrace{\left(\frac{1}{4\tau} + x(1-x)\right)}_{\downarrow \text{ FACTORIZE}}$$

$$-x^2 + x + \frac{1}{4\tau} = 0 \quad \text{FOR} \quad x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1+\frac{1}{\tau}}\right)$$

NOTE

$$x_- < 0 < x < 1 < x_+$$

$$\frac{1}{4\tau} + x(1-x) = \underbrace{(x_+ - x)}_{>0}\underbrace{(x - x_-)}_{>0}$$

NOTE $\quad -x_- = x_+ - 1$

$$\Downarrow$$

DENOTE $\quad c \equiv x_+ = \frac{1}{2}\left(1 + \sqrt{1+\frac{1}{\tau}}\right)$ IN FOLLOWING

$$\therefore\ \Pi_R(q^2) \equiv \frac{\Pi(q^2) - \Pi(0)}{1 - \Pi(0)}$$

$$= \frac{e_R^2}{2\pi^2}\int_0^1 dx\ x(1-x)\left\{\ln\sqrt{4\tau}\,(c-x) + \ln\sqrt{4\tau}\,(c-(1-x))\right\}$$

2 INTEGRALS ARE EQUAL DUE TO SYMMETRY $x \leftrightarrow 1-x$

$$= \frac{e_R^2}{\pi^2}\int_0^1 dx\ x(1-x)\ln\left(\sqrt{4\tau}\,(c-x)\right)$$

$$\downarrow \qquad u \equiv \sqrt{4\tau}\,(c-x) \Rightarrow x = c - \frac{u}{\sqrt{4\tau}}$$

$$= \frac{e_R^2}{\pi^2}\int_{\sqrt{4\tau}\,(c-1)}^{\sqrt{4\tau}\,c} du\ \frac{1}{\sqrt{4\tau}}\left[c(1-c) + \frac{u}{\sqrt{4\tau}}(2c-1) - \frac{u^2}{4\tau}\right]\ln u$$

NOTE: $\quad \sqrt{4\tau}\,(c-1) = \sqrt{4\tau}\,\frac{1}{2}\left(-1 + \sqrt{1+\frac{1}{\tau}}\right)$

$$= \sqrt{4\tau}\,\frac{1}{2}\,\frac{\left(-1+\sqrt{1+\frac{1}{\tau}}\right)\left(1+\sqrt{1+\frac{1}{\tau}}\right)}{\left(1+\sqrt{1+\frac{1}{\tau}}\right)}$$

$$= \sqrt{4\tau}\,\frac{1}{2}\,\frac{1/\tau}{\left(1+\sqrt{1+\frac{1}{\tau}}\right)} = \frac{1}{\sqrt{4\tau}\,c}$$

$$\therefore \quad \Pi_R(q^2) = \frac{e_R^2}{\pi^2} \int_{\frac{1}{\sqrt{4\tau}\,c}}^{\sqrt{4\tau}\,c} du \; \frac{1}{\sqrt{4\tau}} \left[\underbrace{c(1-c)}_{-\frac{1}{4\tau}} + \frac{u}{\sqrt{4\tau}} \underbrace{\left(c+(c-1)\right)}_{\frac{1}{\sqrt{4\tau}}\left(\sqrt{4\tau}\,c + \frac{1}{\sqrt{4\tau}\,c}\right)} - \frac{u^2}{4\tau} \right] \ln u$$

$$= \frac{e_R^2}{\pi^2} \frac{1}{\sqrt{4\tau}} \frac{1}{4\tau} \int_{1/a}^{a} du \left[-1 + u\left(a + \frac{1}{a}\right) - u^2 \right] \ln u$$

WITH $\underline{a \equiv \sqrt{4\tau}\,c}$

$$\boxed{\Pi_R(q^2) = \frac{e_R^2}{\pi^2} \frac{1}{(4\tau)^{3/2}} \int_{1/a}^{a} du \; (u-a)\left(\frac{1}{a} - u\right) \ln u}$$

- MATH HELP

$$I_1 = \int_{1/a}^{a} du \; (u-a)\left(\frac{1}{a} - u\right) \ln u$$

$$= \int_{1/a}^{a} d\left[u \ln u - u\right] \; (u-a)\left(\frac{1}{a} - u\right)$$

$$= -\int_{1/a}^{a} du \left[u \ln u - u\right] \left(\left(\frac{1}{a} - u\right) - (u-a)\right)$$

$$= -\int_{1/a}^{a} du \left\{ \left(\frac{1}{a} - u\right)(u-a) \ln u + (u-a)\left(\frac{1}{a} - u\right) \ln u \right\}$$

$$+ \int_{1/a}^{a} du \left\{ -a\left(\frac{1}{a} - u\right) \ln u + \frac{1}{a}(u-a) \ln u \right\}$$

$$+ \int_{1/a}^{a} du \; u\left(\frac{1}{a} + a - 2u\right)$$

$$3I_1 = I_2 + \left[\frac{1}{2}\left(\frac{1}{a}+a\right)\left(a^2-\frac{1}{a^2}\right) - \frac{2}{3}\left(a^3-\frac{1}{a^3}\right)\right]$$

$$\boxed{I_1 = \frac{1}{3}I_2 + \frac{1}{18}\left(a-\frac{1}{a}\right)\left[2-a^2-\frac{1}{a^2}\right]}$$

WITH

$$I_2 \equiv \int_{1/a}^{a} du \left\{-a\left(\frac{1}{a}-u\right) + \frac{1}{a}(u-a)\right\} \ln u$$

$$= \int_{1/a}^{a} d\left[u\ln u - u\right] \left\{-a\left(\frac{1}{a}-u\right) + \frac{1}{a}(u-a)\right\}$$

$$= -a^2\left(\frac{1}{a}-a\right)\left[\ln a - 1\right] - \frac{1}{a^2}\left(\frac{1}{a}-a\right)\left[-\ln a - 1\right]$$

$$- \int_{1/a}^{a} du \left[u\ln u - u\right]\left(a+\frac{1}{a}\right)$$

$$= \ln a \left(\frac{1}{a}-a\right)\left(\frac{1}{a^2}-a^2\right) + \left(\frac{1}{a}-a\right)\left(\frac{1}{a^2}+a^2\right)$$

$$- \int_{1/a}^{a} du \left[-a\left(\frac{1}{a}-u\right) + \frac{1}{a}(u-a)\right] \ln u$$

$$+ \int_{1/a}^{a} du \left[-1 \quad -1\right] \ln u + \frac{1}{2}\left(a+\frac{1}{a}\right)\left(a^2-\frac{1}{a^2}\right)$$

$$2I_2 = \ln a \left(\frac{1}{a}-a\right)^2\left(\frac{1}{a}+a\right) - 2\left(\frac{1}{a}+a\right)\ln a$$

$$+ \left(\frac{1}{a}-a\right)\left\{-2 + \frac{1}{a^2}+a^2 - \frac{1}{2}\left(a^2+\frac{1}{a^2}+2\right)\right\}$$

$$\boxed{I_2 = \frac{1}{2}\left(\frac{1}{a}+a\right)\left(\frac{1}{a^2}+a^2-4\right)\ln a + \frac{1}{4}\left(\frac{1}{a}-a\right)\left(\frac{1}{a^2}+a^2-6\right)}$$

$$\therefore\quad \Pi_R(q^2) = \frac{e_R^2}{36\pi^2}\,\frac{1}{(4\tau)^{3/2}} \cdot \left\{ 6\left(\frac{1}{a}+a\right)\left(\frac{1}{a^2}+a^2-4\right)\ln a + 3\left(\frac{1}{a}-a\right)\left(\frac{1}{a^2}+a^2-6\right) + 2\left(\frac{1}{a}-a\right)\left(\frac{1}{a^2}+a^2-2\right)\right\}$$

$$\Pi_R(q^2) = \frac{e_R^2}{36\pi^2}\,\frac{1}{(4\tau)^{3/2}} \cdot \left\{ 6\left(\frac{1}{a}+a\right)\left(\frac{1}{a^2}+a^2-4\right)\ln a + \left(\frac{1}{a}-a\right)\left(\frac{5}{a^2}+5a^2-22\right)\right\}$$

$$\hookrightarrow\quad a+\frac{1}{a} = \sqrt{4\tau}\,c + \frac{1}{\sqrt{4\tau}\,c} = \sqrt{4\tau}\,(2c-1) = \sqrt{4\tau}\,\sqrt{1+\frac{1}{\tau}}$$

$$\hookrightarrow\quad a-\frac{1}{a} = \sqrt{4\tau}\,c - \frac{1}{\sqrt{4\tau}\,c} = \sqrt{4\tau} = \sqrt{\frac{Q^2}{m^2}}$$

$$\hookrightarrow\quad a^2+\frac{1}{a^2} = 4\tau\left(c^2+(c-1)^2\right) = 2\tau\left((2c-1)^2+1\right) = 2\tau\left(1+\frac{1}{\tau}+1\right) = 2\tau\left(2+\frac{1}{\tau}\right)$$

$$\therefore\quad \Pi_R(q^2) = \frac{e_R^2}{36\pi^2}\left\{ 6\sqrt{1+\frac{1}{\tau}}\left(1-\frac{1}{2\tau}\right)\ln\left(\sqrt{4\tau}\,c\right) - 5 + \frac{3}{\tau}\right\}$$

$$\downarrow\qquad \alpha \equiv \frac{e_R^2}{4\pi} = \frac{1}{137}\quad,\quad Q^2 = -q^2$$

$$\boxed{\Pi_R(-Q^2) = \frac{\alpha}{9\pi}\left\{ -5 + \frac{12m^2}{Q^2} + 6\left(1-\frac{2m^2}{Q^2}\right)\sqrt{1+\frac{4m^2}{Q^2}}\,\ln\left[\sqrt{\frac{Q^2}{4m^2}}\left(1+\sqrt{1+\frac{4m^2}{Q^2}}\right)\right]\right\}}$$

NOTE FOR $Q^2 \ll m^2$

$$\Pi_R(-Q^2) = \frac{\alpha}{15\pi}\frac{Q^2}{m^2} + O(Q^4)$$

REGULAR FOR $Q^2 \to 0$

- POTENTIAL IN COORDINATE SPACE

$$\tilde{V}_U(q^2) = -\frac{e_R^2}{Q^2}\cdot\Pi_R(-Q^2) = \frac{e_R^2}{q^2}\Pi_R(q^2)$$

$$\Downarrow$$

$$V_U(r) = \int\frac{d^3\bar{q}}{(2\pi)^3}\, e^{i\bar{q}\bar{r}}\, \tilde{V}_U(-Q^2)$$

$Q^2 = -q^2$

WITH $Q^2 \simeq \bar{q}^2$ $(q^0 = 0)$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dQ\, Q^2\left(-\frac{e_R^2}{Q^2}\right)\Pi_R(-Q^2)\int_{-1}^{1} dx\, e^{iQrx}$$

$$= -\frac{\alpha}{\pi}\int_0^\infty dQ\,\frac{1}{iQr}\,\Pi_R(-Q^2)\left[e^{iQr} - e^{-iQr}\right]$$

$$= -\frac{\alpha}{r}\frac{1}{i\pi}\int_{-\infty}^{+\infty} dQ\,\frac{1}{Q}\,\Pi_R(-Q^2)\, e^{iQr}$$

$$\downarrow$$

PERFORM ANALYTIC CONTINUATION TO COMPLEX Q-PLANE

AS $r > 0$ $\Rightarrow$ INTEGRAL OVER SEMI-CIRCLE IN UPPER COMPLEX Q-PLANE VANISHES.

SINGULARITIES OF $\Pi_R(-Q^2)$ IN COMPLEX PLANE?

↳ $Q = 0$: NO SINGULARITY

$$\frac{1}{Q}\,\Pi_R(-Q^2) \xrightarrow[Q\to 0]{} \sim Q \to 0$$

↳ LOG FUNCTION HAS CUT ALONG POSITIVE IMAGINARY AXIS

FOR $Q = iq$

$$\ln\left[\sqrt{\frac{Q^2}{4m^2}}\left(1+\sqrt{1+\frac{4m^2}{Q^2}}\right)\right]$$

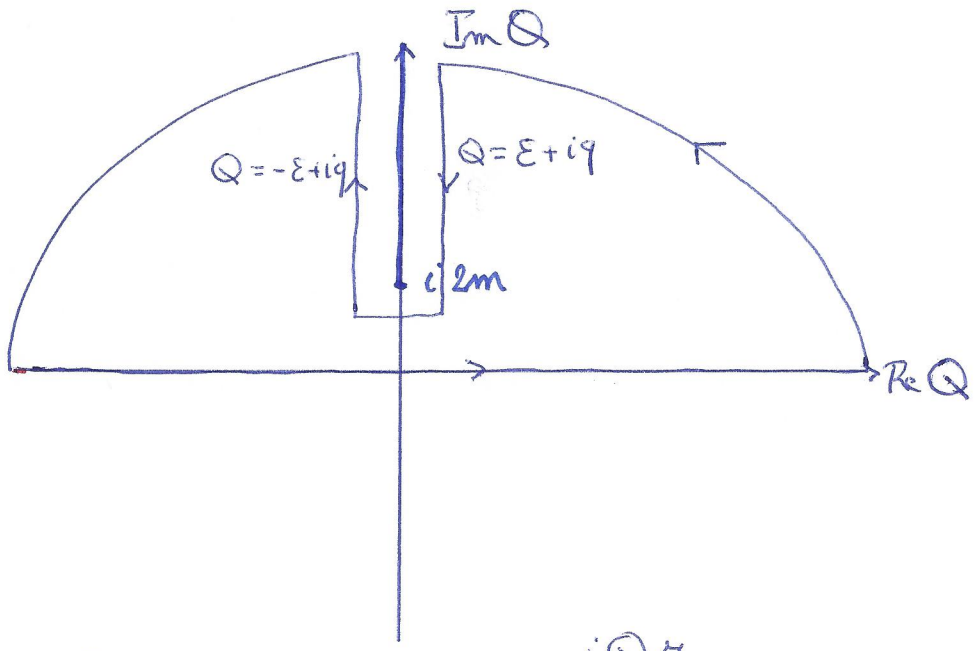
$$= \frac{1}{2}\ln \frac{\left(1+\sqrt{1+\frac{4m^2}{Q^2}}\right)^2}{\left(1+\sqrt{1+\frac{4m^2}{Q^2}}\right)\left(-1+\sqrt{1+\frac{4m^2}{Q^2}}\right)}$$

$$= \frac{1}{2}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{q^2}}}{-1+\sqrt{1-\frac{4m^2}{q^2}}}\right]$$

CUT FOR $q^2 > 4m^2$

FOR $q^2 > 4m^2$ ALONG IM AXIS

IMAGINARY PART OF LOG IS $\frac{1}{2}(-i\pi) \Rightarrow -\frac{\pi}{2}$



$$\int_{-\infty}^{+\infty} dQ \; \frac{1}{Q} \; \Pi_R(-Q^2)\, e^{iQr}$$

$$= -\int_{i\infty}^{i2m} d(iq) \; \frac{e^{-qr}}{iq} \; \Pi_R(q^2 - i\varepsilon) - \int_{i2m}^{+i\infty} d(iq) \frac{e^{-qr}}{iq} \; \Pi_R(q^2 + i\varepsilon)$$

$$= + \int_{2m}^{\infty} dq \; \frac{e^{-qr}}{q} \; \underbrace{\left\{ \Pi_R(q^2 - i\varepsilon) - \Pi_R(q^2 + i\varepsilon) \right\}}$$

DISCONTINUITY (IMAGINARY PART) OF SELF-ENERGY Π_R ACROSS THE CUT FOR $q^2 > 4m^2$

$$-2i \, \mathrm{Im}\, \Pi_R(q^2)$$

$$\Rightarrow \quad \mathrm{Im}\, \Pi_R(q^2) = \frac{\alpha}{9\pi} 6 \left(1 + \frac{2m^2}{q^2}\right) \sqrt{1 - \frac{4m^2}{q^2}} \cdot \left(-\frac{\pi}{2}\right) \cdot \Theta(q^2 - 4m^2)$$

$$\boxed{\mathrm{Im}\, \Pi_R(q^2) = -\frac{\alpha}{3} \left(1 + \frac{2m^2}{q^2}\right) \sqrt{1 - \frac{4m^2}{q^2}}\, \Theta(q^2 - 4m^2)}$$

$$\therefore\quad V_U(r) = -\frac{\alpha}{r}\cdot\frac{2\alpha}{3\pi}\int_{2m}^{\infty} dq\,\frac{e^{-qr}}{q}\left(1+\frac{2m^2}{q^2}\right)\sqrt{1-\frac{4m^2}{q^2}}$$

$$\downarrow \qquad \tilde{q} = \frac{q}{2m}$$

$$\boxed{V_U(r) = -\frac{\alpha}{r}\cdot\frac{2\alpha}{3\pi}\int_{1}^{\infty} d\tilde{q}\; e^{-2mr\tilde{q}}\left(\frac{2\tilde{q}^2+1}{2\tilde{q}^4}\right)\sqrt{\tilde{q}^2-1}}$$

ATTRACTIVE CORRECTION TO COULOMB POTENTIAL DUE TO PHOTON VACUUM POLARIZATION

- EXPANSION FOR $r \gg \frac{1}{2m}$ LARGE RELATIVE TO COMPTON WAVELENGTH OF e^-

$$\frac{1}{2m_e} \simeq \frac{197\ \text{MeV fm}}{1\ \text{MeV}} \approx 200\ \text{fm}$$

INTEGRAL WILL BE DOMINATED BY REGION AROUND $\tilde{q} = 1$

$\tilde{q} = 1 + \frac{x}{2m}$ $\quad$ x : small quantity

$$V_U(r) \simeq -\frac{2\alpha^2}{3\pi}\cdot\frac{1}{r}\,e^{-2mr}\int_0^{\infty} dx\,\frac{e^{-xr}}{2m\sqrt{m}}\sqrt{x}\,\frac{3}{2}\left[1+O\left(\frac{1}{2mr}\right)\right]$$

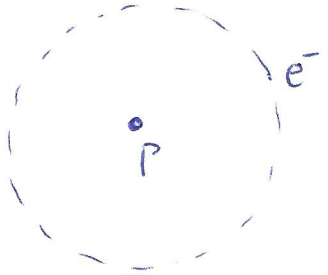
$$\downarrow\ \tilde{x} = xr$$

$$= -\frac{\alpha^2}{\pi}\cdot\frac{e^{-2mr}}{2m^{3/2}r}\,\frac{1}{r^{3/2}}\underbrace{\int_0^{\infty} d\tilde{x}\; e^{-\tilde{x}}\,\tilde{x}^{1/2}}_{\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}}\left[1+O\left(\frac{1}{2mr}\right)\right]$$

$$\boxed{V_U(r) \simeq -\frac{\alpha}{r}\cdot\frac{\alpha}{4\sqrt{\pi}}\cdot\frac{e^{-2mr}}{(mr)^{3/2}}\left[1+O\left(\frac{1}{2mr}\right)\right]}$$

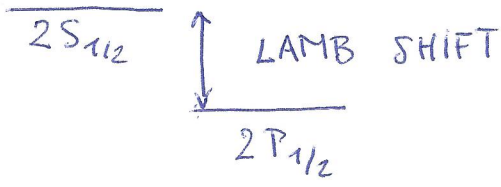
⇒ LAMB SHIFT

- FOR ORDINARY H-ATOM

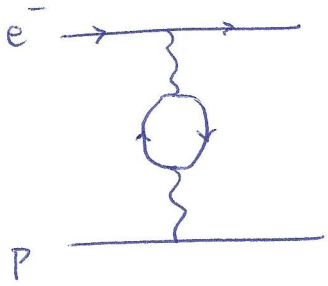


IN DIRAC THEORY

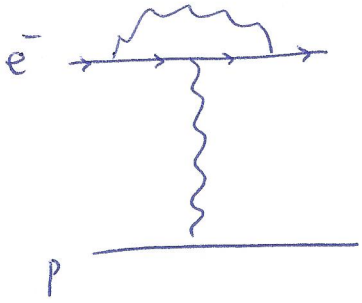
IN QFT



LAMB SHIFT HAS SEVERAL CONTRIBUTIONS



VACUUM POLARIZATION



e^- SELF-ENERGY
(DOMINATES FOR ORDINARY H-ATOM)

↳ WAVE FUNCTION

$$\psi_{2S}(\vec{r}) = \psi_{2S}(0)\left(1-\frac{r}{2a}\right)e^{-\frac{r}{2a}}$$

BOHR RADIUS $\| \; a = \frac{1}{\mu\alpha}$, $\mu = \frac{m_e m_p}{m_e + m_p}$, $\alpha = \frac{1}{137}$

REDUCED MASS

FOR ORDINARY H-ATOM $\mu \simeq m_e$

$$a = \frac{1}{\mu\alpha} \simeq \frac{137}{0.5\,\text{MeV}} \simeq 5\,10^4\ \text{fm}$$

NORMALIZATION

$$1 = \int d^3\bar{r}\, |\psi_{2S}(\bar{r})|^2$$

$$= |\psi_{2S}(0)|^2\, 4\pi \int_0^\infty dr\, r^2 \left(1 - \frac{r}{2a}\right)^2 e^{-\frac{r}{a}}$$

$$\downarrow \quad x = r/a$$

$$= |\psi_{2S}(0)|^2\, 4\pi\, a^3 \underbrace{\int_0^\infty dx\, e^{-x}\, x^2 \left(1 - x + \frac{1}{4}x^2\right)}$$

$$\Gamma(3) - \Gamma(4) + \frac{1}{4}\Gamma(5)$$

$$= |\psi_{2S}(0)|^2\, 4\pi\, 2a^3 \qquad \frac{\Gamma(3)}{2}$$

$$|\psi_{2S}(0)| = \frac{1}{\sqrt{4\pi}} \left(\frac{1}{2a^3}\right)^{1/2}$$

↳ SHIFT OF 2S LEVEL DUE TO VACUUM POLARIZATION

$$\Delta E_{2S} = \langle \psi_{2S} | V_U(r) | \psi_{2S} \rangle$$

$$= \int d^3\bar{r}\, \psi^*_{2S}(\bar{r})\, V_U(r)\, \psi_{2S}(\bar{r})$$

$$= \left(\frac{1}{2a^3}\right) \int_0^\infty dr\, r^2\, e^{-\frac{r}{a}} \left(1 - \frac{r}{2a}\right)^2 \underbrace{V_U(r)}$$

↓ GENERAL EXPRESSION

$$= -\frac{\alpha^2}{\pi}\frac{2}{3} \int_1^\infty d\tilde{q} \left(\frac{2\tilde{q}^2 + 1}{2\tilde{q}^4}\right) \sqrt{\tilde{q}^2 - 1}$$

$$\cdot \left(\frac{1}{2a^3}\right) \cdot \underbrace{\int_0^\infty dr\, e^{-\frac{r}{a}}\, r^2 \left(1 - \frac{r}{2a}\right)^2 \frac{e^{-2m_e r \tilde{q}}}{r}}$$

$$\int_0^\infty dr\, e^{-r\left(\frac{1}{a} + 2m_e\tilde{q}\right)}\, r \left(1 - \frac{r}{a} + \frac{1}{4}\frac{r^2}{a^2}\right)$$

$$\tilde{r} = 2m_e r$$

$$\Delta E_{2S} = -\frac{\alpha^2}{\pi}\cdot\frac{2}{3}\,\frac{1}{(2m_e)^2}\int_1^\infty d\tilde{q}\left(\frac{2\tilde{q}^2+1}{2\tilde{q}^4}\right)\sqrt{\tilde{q}^2-1}$$

$$\cdot\left(\frac{1}{2a^3}\right)\cdot\int_0^\infty d\tilde{r}\; e^{-\tilde{r}(\tilde{q}+K)}\,\tilde{r}\left(1-K\tilde{r}+\frac{1}{4}K^2\tilde{r}^2\right)$$

WITH $\boxed{K \equiv \frac{1}{2m_e a} = \frac{m\alpha}{2m_e}}$

$$\downarrow \qquad x \equiv \tilde{r}(\tilde{q}+K)$$

$$\Delta E_{2S} = -\frac{\alpha^2}{\pi}\,\frac{2}{3}\,\frac{1}{(2m_e)}\,\frac{1}{2(2a^3)}\int_1^\infty d\tilde{q}\left(\frac{2\tilde{q}^2+1}{2\tilde{q}^4}\right)\frac{\sqrt{\tilde{q}^2-1}}{(\tilde{q}+K)^2}$$

$$\cdot\int_0^\infty dx\; e^{-x}\left(x-\frac{K}{\tilde{q}+K}x^2+\frac{1}{4}\frac{K^2}{(\tilde{q}+K)^2}x^3\right)$$

$$\Gamma(2)-\frac{K}{\tilde{q}+K}\Gamma(3)+\frac{1}{4}\frac{K^2}{(\tilde{q}+K)^2}\Gamma(4)$$

$$\downarrow$$

$$\frac{1}{(\tilde{q}+K)^2}\underbrace{\left\{(\tilde{q}+K)^2-2K(\tilde{q}+K)+\frac{3}{2}K^2\right\}}_{\tilde{q}^2+\frac{K^2}{2}}$$

$$\boxed{\Delta E_{2S} = -\frac{m\alpha^3}{3\pi}\cdot\left(\frac{m\alpha}{2m_e}\right)^2\int_1^\infty d\tilde{q}\left(\frac{2\tilde{q}^2+1}{2\tilde{q}^4}\right)\left(\frac{\tilde{q}^2+K^2/2}{(\tilde{q}+K)^4}\right)\sqrt{\tilde{q}^2-1}}$$

$$= -\frac{m\alpha^3}{3\pi}\left(\frac{m\alpha}{2m_e}\right)^2\cdot I(K)$$

↳ FOR ORDINARY H-ATOM

$$K = \frac{\mu\alpha}{2m_e} \simeq \frac{m_e\alpha}{2m_e} = \frac{\alpha}{2} \simeq 0.0036$$

$$\downarrow$$

$$I(K=0) = \int_1^\infty d\tilde{q} \left(\frac{2\tilde{q}^2+1}{2\tilde{q}^6}\right)\sqrt{\tilde{q}^2-1}$$

$$= \frac{4}{10}$$

$$I(K=0.0036) = 0.396 = \frac{4}{10}\cdot(0.99)$$

$$\Delta E_{2S} = -\frac{m_e\alpha^3}{3\pi}\cdot\frac{\alpha^2}{4}\cdot\frac{4}{10}\cdot(0.99)$$

$$\boxed{\Delta E_{2S} = -\frac{m_e\alpha^5}{30\pi}\cdot(0.99)} = -1.1\ 10^{-7}\ eV$$

(BUT ONLY 2.5% OF TOTAL LAMB SHIFT FOR ORDINARY H)

NOTE: FOR $K \ll 1$ ⇒ δ-FUNCTION APPROXIMATION VERY GOOD

$$\downarrow$$

$$\underbrace{\frac{1}{2m_e}}_{\text{RANGE OF UEHLING POT}} \ll \underbrace{\frac{1}{\mu\alpha}}_{\text{SIZE OF H-ATOM}}$$

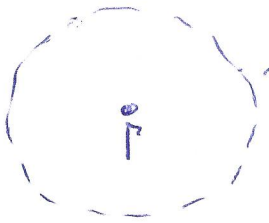
$$V_U(r) \simeq -\frac{4}{15}\frac{\alpha^2}{m_e^2}\delta^3(\vec{r})$$

$$\Delta E_{2S} = -\frac{4}{15}\frac{\alpha^2}{m_e^2}\underbrace{|\psi_{2S}(0)|^2}_{\frac{1}{4\pi}\cdot\frac{(m_e\alpha)^3}{2}} = -\frac{m_e\alpha^5}{30\pi}$$

∴ δ-FUNCTION POTENTIAL GIVES RESULT UP TO 1%

• MUONIC H-ATOM

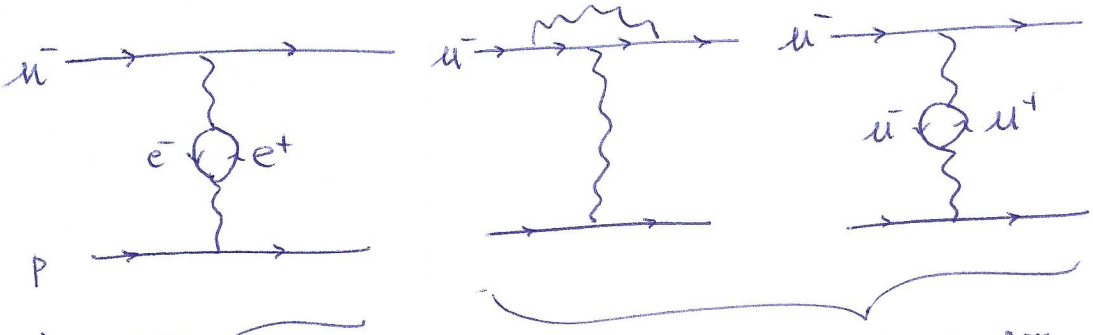
↳



$m_\mu \simeq 105$ MeV

SIZE : $a = \frac{1}{m_\mu \alpha} \simeq \frac{197}{5} \cdot (137)$ fm $\simeq 280$ fm

FOR MUONIC ATOM : LAMB SHIFT CORRECTIONS



DOMINATES P_0

GIVE CORRECTIONS AT 1% LEVEL DUE TO HEAVY MUON MASS

⇓

DIAGRAMS WITH MUON PROPAGATORS ARE SUPPRESSED

↳ VACUUM POLARIZATION CONTRIBUTION

$$\Delta E_{2S}^{VP} = -\frac{m_\mu \alpha^3}{3\pi} \cdot K^2 \, I(K)$$

$$K = \frac{m_\mu \alpha}{2 m_e} \simeq 0.678$$

$$I(0.678) \simeq 0.13$$

$$\boxed{\Delta E_{2S}^{VP} \simeq -\ 219.6 \text{ meV}}$$

$$\Delta E_{2P}^{VP} = -\ 14.6 \text{ meV}$$

VACUUM POLARIZATION CONTR. TO LAMB SHIFT FOR μ-H

$$\boxed{\Delta E_{LS}^{VP} = \Delta(E_{2P} - E_{2S}) = +\ 205 \text{ meV}}$$

↳ FINITE PROTON SIZE CORRECTION

AS SIZE OF MUONIC ATOM $\frac{1}{m\alpha} \sim 280$ fm

A PRECISE MEASUREMENT OF ΔE_{LS} IS SENSITIVE TO FINITE CHARGE DISTRIBUTION OF PROTON ⟹ SIZE ~ 1 fm

⇝ COULOMB POTENTIAL POINT PROTON

$$\tilde{V}_C(q^2) = \frac{e^2}{q^2}$$

⇝ POTENTIAL INCLUDING FINITE SIZE OF PROTON

$$\tilde{V}(q^2) = \frac{e^2}{q^2} G_E(q^2)$$

WITH $G_E(q^2)$: ELECTRIC FORM FACTOR OF PROTON

$$G_E(q^2) = \underbrace{1}_{\uparrow} + \frac{1}{6}\underbrace{\langle r^2 \rangle}_{\downarrow} q^2 + O(q^4)$$

$G_E(0)$ CHARGE

RMS RADIUS OF P CHARGE DISTRIBUTION

∴ FINITE SIZE (FS) GIVES CORRECTION TO COULOMB POTENTIAL

$$\tilde{V}^{FS}(q^2) = \frac{e^2}{q^2} \cdot \frac{1}{6}\langle r^2 \rangle q^2 = \frac{e^2}{6}\langle r^2 \rangle$$

CONSTANT

$$\Downarrow$$

$$\boxed{V^{FS}(r) = \frac{4\pi\alpha}{6}\langle r^2 \rangle \delta^3(\vec{r})}$$

⇝ FINITE SIZE CORRECTION TO 2S LEVEL

$$\Delta E_{2S}^{FS} = \frac{4\pi\alpha}{6} \langle r^2 \rangle \left| \psi_{2S}(0) \right|^2$$

$$= \frac{4\pi\alpha}{6} \langle r^2 \rangle \frac{1}{4\pi} \frac{1}{2a^3}$$

$$= \frac{\mu^3 \alpha^4}{12} \langle r^2 \rangle$$

$$\boxed{\Delta E_{2S}^{FS} \simeq (5 \text{ meV}) \cdot \langle r^2 (\text{in fm}^2) \rangle}$$

$\downarrow$ FOR $\sqrt{\langle r^2 \rangle} \simeq 0.84$ fm

$$\underline{\Delta E_{2S}^{FS} \simeq 3.7 \text{ meV}} \quad \Rightarrow \Delta E_{LS}^{FS} = \Delta (E_{2P} - E_{2S})^{FS} = -3.7 \text{ meV}$$

NOTE: 2013 EXPERIMENT ON ΔE_{LS} IN μH REACHED PRECISION OF 2.3 meV ∇_a