Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 9

20.06.2022

Exercise 1. (100+25 points): Feynman integrals

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(25 points) The following identity was proven in lectures:

$$\frac{1}{A_1 A_2} = \int_0^1 \frac{1}{\left[x_1 A_1 + (1 - x_1)\, A_2\right]^2}\, dx_1$$

Usually it is more convenient to write this formula as a double integral (symmetric form):

$$\frac{1}{A_1 A_2} = \int_0^1 \int_0^1 \delta\left(x_1 + x_2 - 1\right) \frac{1}{\left[x_1 A_1 + x_2 A_2\right]^2}\, dx_1 dx_2$$

Differentiate it $n-1$ times to get the following identity:

$$\frac{1}{A_1 A_2^n} = \int_0^1 \int_0^1 \delta\left(x_1 + x_2 - 1\right) \frac{n x_2^{n-1}}{\left[x_1 A_1 + x_2 A_2\right]^{n+1}}\, dx_1 dx_2$$

(b)(25 points) Using the result from the previous part, prove the general formula:

$$\frac{1}{A_1 ... A_k} = \int_0^1 ... \int_0^1 \delta\left(x_1 + ... + x_k - 1\right) \frac{(k-1)!}{\left[x_1 A_1 + ... x_k A_k\right]^k}\, dx_1 ... dx_k$$

Hint: do this by induction - you know that formula is valid for a certain n (=2), then prove it works for $n+1$.

(c)(25 points) Consider the integral:

$$I = \int \frac{1}{\left(p^2 - \Delta + i\varepsilon\right)^n} \frac{d^D p}{(2\pi)^D}$$

In the lectures this integral was calculated using the Wick rotation with the assumption that $\Delta > 0$. Now show that the Wick rotation method still works for $\Delta < 0$.
Hint: note that poles in this case remain in the same quarters of the integration plane.

(d)(25 points) Explain why two following integrals are zero:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 - \Delta + i\epsilon)^n} = 0$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\sigma}{(k^2 - \Delta + i\epsilon)^n} = 0$$

And prove the following identities:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta + i\epsilon)^n} = -\frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \frac{\Gamma\left(n - 1 - \frac{D}{2}\right)}{2\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n-1-D/2}}$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\sigma k^\rho}{(k^2 - \Delta + i\epsilon)^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(n - 2 - \frac{D}{2}\right)}{\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n-2-D/2}} \frac{(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu})}{4}$$

(e*)(Advanced level problem for those who are interested - 25 points) Using previous results, also prove:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2 + 2(pk) - m^2 + i\varepsilon)^n} = \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{(p^2 + m^2 - i\varepsilon)^{n-D/2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)}$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + 2(pk) - m^2 + i\varepsilon)^n} = -\frac{i p^\mu}{(4\pi)^{D/2}} \frac{(-1)^n}{(p^2 + m^2 - i\varepsilon)^{n-D/2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)}$$

Additionally:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 + 2(pk) - m^2 + i\varepsilon)^n} =$$
$$= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{(p^2 + m^2 - i\varepsilon)^{n-D/2}} \frac{1}{\Gamma(n)} \left[p^\mu p^\nu \Gamma\left(n - \frac{D}{2}\right) - \frac{1}{2} g^{\mu\nu} \left(p^2 + m^2\right) \Gamma\left(n - 1 - \frac{D}{2}\right)\right]$$

One more:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\rho}{(k^2 + 2(pk) - m^2 + i\varepsilon)^n} =$$
$$= -\frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{(p^2 + m^2 - i\varepsilon)^{n-D/2}} \frac{1}{\Gamma(n)} \times$$
$$\times \left[p^\mu p^\nu p^\rho \Gamma\left(n - \frac{D}{2}\right) - \frac{1}{2} \left(g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu + g^{\rho\nu} p^\mu\right) \left(p^2 + m^2\right) \Gamma\left(n - 1 - \frac{D}{2}\right)\right]$$

And the last one:

$$
\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\rho k^\sigma}{\left(k^2 + 2\,(pk) - m^2 + i\varepsilon\right)^n} =
$$

$$
= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{\left(p^2 + m^2 - i\varepsilon\right)^{n-D/2}} \frac{1}{\Gamma\,(n)} \times \left[p^\mu p^\nu p^\rho p^\sigma \Gamma\left(n - \frac{D}{2}\right) - \right.
$$

$$
- \frac{1}{2} \left(g^{\mu\nu} p^\rho p^\sigma + g^{\mu\rho} p^\nu p^\sigma + g^{\mu\sigma} p^\nu p^\rho + g^{\nu\rho} p^\mu p^\sigma + g^{\nu\sigma} p^\rho p^\mu + g^{\rho\sigma} p^\nu p^\mu\right) \left(p^2 + m^2\right) \Gamma\left(n - 1 - \frac{D}{2}\right) +
$$

$$
\left. + \Gamma\left(n - 2 - \frac{D}{2}\right) \left(p^2 + m^2\right)^2 \frac{\left(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu}\right)}{4} \right]
$$

Hint: it is way easier than it may seem. Make a replacement $k + p \to k'$.

Literature

1. Problem Book Quantum Field Theory, Radovanovic V. - part 11.
2. See lecture notes on renormalization.