Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 5

23.05.2022

Exercise 1. (100+25 points): Photon field + Scattering in scalar theory

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(25 points) Consider the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Em} + \mathcal{L}_{G.F.} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$$

where ξ is a real constant. Derive the equations of motion and invert the result to obtain the general photon Feynman propagator.
Hint: In momentum space the equations of motion contain no any other vectors or tensors except for $g^{\mu\nu}$ and k^μ. Thus the propagator can be parameterized as:

$$D^{\mu\nu} = A(k^2) g^{\mu\nu} + B(k^2) k^\mu k^\nu$$

Note: Physical results do not depend on the choice of ξ, because this term goes zero anyway after gauge fixing, thus for simplicity it is often set to be 1. Sometimes the choice of ξ is called by abuse of language a choice of gauge, $\xi = 1$ refers to the Feynman "gauge".

(b)(25 points) Consider a state $|\Psi_T\rangle$ which only contains transverse photons. Futhermore, construct a state $|\Psi'_T\rangle$ as:

$$|\Psi'_T\rangle = \left\{1 + \alpha \left[a^\dagger(\vec{k}, 3) - a^\dagger(\vec{k}, 0)\right]\right\} |\Psi_T\rangle,$$

with α a constant. Show that replacing $|\Psi_T\rangle$ by $|\Psi'_T\rangle$ corresponds to a gauge transformation:

$$\langle\Psi'_T|A^\mu(x)|\Psi'_T\rangle = \langle\Psi_T|A^\mu(x) + \partial^\mu\Lambda|\Psi_T\rangle\,,$$

where Λ is given by:

$$\Lambda(x) = \mathrm{Re}\left(i\alpha \frac{\sqrt{2}}{\omega_k^{3/2}} e^{-ik\cdot x}\right).$$

Note: $\partial_\mu \partial^\mu \Lambda = 0$, which means this is a transformation within Lorentz gauge.

(c)(50 points) Considering the interaction Lagrangian for scalar fields

$$\mathcal{L}_1 = -\frac{\lambda}{4!}\phi^4,$$

and the Dyson Expansion of the S-Matrix:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots \int d^4x_n T\left\{\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)\right\}.$$

Calculate the second order ($n = 2$) S-matrix element for a process of 2 initial bosons (of momenta p_1 and p_2) going to 4 final ones (of momenta p_3, p_4, p_5 and p_6) and draw the diagrams which arise from it (at least 2 re-orderings of the external fields).

(d*)(Advanced level problem for those who are interested - 25 points) Consider the abelian Chern-Simons action ($D = 1 + 2$):

$$S = \frac{\eta}{4\pi}\varepsilon^{\mu\nu\rho} \int A_\mu \partial_\nu A_\rho \, d^3x$$

Assuming field A vanishes at infinity, prove that this theory is gauge invariant, derive the equations of motion and check that energy-momentum tensor is zero.
Tip: This theory belongs to the class of topological field theories and appears to be way more non-trivial than is may seem at the first glance. The fact that energy-momentum tensor is zero actually means that this theory doesn't depend on metric at all. Such theories attract a lot of attention when it comes to quantum gravity.

Literature

1. See lecture notes on photon field.
2. Quantum Field Theory. Zuber J.-B., Itzykson C. (section 3.2.2). This book contains a lot of wisdom, but be careful about the notation and possible typos.