Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 4

16.05.2022

Exercise 1. (100 points): Dirac field

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(25 points) For a Dirac field, the transformations:

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger\prime}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5},$$

where α is an arbitrary real parameter, are called chiral phase transformations. Show that the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J^\mu_A \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

(b)(25 points) Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Hence, the Lagrangian density $\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L$ describes massless fermions with negative helicity and massless anti-fermions with positive helicity only. This field is called the Weyl field and can be used to describe neutrinos as far as the latter can be considered as massless.
Hint: use Weyl representation for gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

(c)(50 points) Find the gamma matrices in space-time $D = 1 + 1$, $D = 1 + 2$ and $D = 1 + 4$ in an arbitrary representation.

(d*)(Advanced level problem for those who are interested - 25 points) Two-dimensional massless QED (usually called Schwinger model) appears to be an extremely valuable theory with a lot of non-trivial consequences. For example, it

exhibits confinement, conformal anomaly, axial anomaly, bosonization and many other properties.
Let's for simplicity fix the gamma matrices in the following form (standard $2D$ representation):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In $2D$ space-time the "fifth" gamma matrix is conveniently defined by the formula:

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

As a first step check that ψ_L (ψ_R) in $D = 2$ stands for particles which move to the left (to the right). Secondly, prove that in $D = 2$ the numbers of both left and right particles are fixed and can't be changed.
Hint: it follows from the fact that there are two conserved quantities from two symmetries, j^0 and j^0_A. But note that production of $L\bar{L}$ ($R\bar{R}$) pairs, i.e. left (right) fermion and antifermion, does not change the number of left particles, because their contributions cancel each other.

Literature

1. Classical Theory of Gauge Fields. Rubakov V., Wilson S.S (chapters 14 and 15).
Be careful about the notation - it is different from lectures.