Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 2

02.02.2022

Exercise 1 (100+25 points): Dirac field

(0)(0 points) How much time did you spend in solving this exercise sheet?

(a)(50 points) Without using an explicit representation of the gamma matrices, prove the following identities:

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu}, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho] &= 4(g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu}), \\
\gamma_5 &= \frac{i}{4!} \epsilon_{\mu\nu\sigma\rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho, \\
\gamma_5^2 &= 1, \\
[\gamma_5, \gamma^\mu]_+ &= 0, \\
\mathrm{Tr}[\gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho \gamma_5] &= 4i\epsilon^{\mu\nu\sigma\rho}, \\
\mathrm{Tr}[\gamma^{\mu_1} ... \gamma^{\mu_n}] &= 0, \text{ if } n \text{ is odd,}
\end{aligned}$$

Where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon_{0123} = 1$.

(b)(50 points) Real-valued Lagrangian for spinor field is:

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi\right) - V\left(\bar{\psi}\psi\right)$$

It is invariant under Lorentz transformations of general form.

For simplicity consider the potential $V = m\bar{\psi}\psi$. Identify the equations of motion. Find the energy momentum-tensor (use either Noether theorem or metric variation) and the conserved current. Check their conservation and make sure that $T^{\mu\nu}$ is symmetric.

(c*)(Advanced level problem for those who are interested - 25 points) Dirac equation in Hamiltonian form is:

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi$$
$$\hat{H} = \alpha_i \hat{p}_i + \beta m$$

Matrices α and β have following properties:

$$[\alpha_i, \beta]_+ = 0$$
$$[\alpha_i, \alpha_k]_+ = 2\delta_{ik}$$
$$\beta = \gamma^0$$
$$\beta\alpha_i = \gamma_i$$

By analogy with classical mechanics we can introduce the velocity operator:

$$\hat{v}_i = \frac{\partial \hat{H}}{\partial \hat{p}_i} = \alpha_i$$

And the magnetic moment operator:

$$\hat{\mu}_i = \frac{e}{2}\left[\hat{\vec{\mathbf{r}}} \times \hat{\vec{\mathbf{v}}}\right]_i$$

Prove that the expectation value of this operator in the stationary state (i.e. in the state with defined energy) is:

$$\langle\psi|\hat{\mu}_i|\psi\rangle = \frac{e}{2m\gamma}\langle\psi|\hat{L}_i + 2\hat{S}_i|\psi\rangle$$

Where L_i is the orbital momentum, S_i is the spin and γ is the Lorentz-factor:

$$\gamma = \frac{1}{\sqrt{1-v^2}} \tag{1}$$

Hint: the proof can be simplified if you use the following identity:

$$\langle\psi|\hat{\mu}_i|\psi\rangle = \frac{1}{2E}\langle\psi|[\hat{H}, \hat{\mu}_i]_+|\psi\rangle$$

Literature

1. Gamma matrices on Wikipedia
2.The energy–momentum tensor(s) in classical gauge theories. Daniel N. Blaschke, François Gieres, Méril Reboud, Manfred Schweda, 2016.