Theoretical Physics 6a (QFT): SS 2022
Exercise sheet 1

25.04.2022

Exercise 1 (100+25 points): Complex scalar theory

Let ϕ denote the complex scalar field. Consider the following Lagrangian density with potential energy V in metric $(+,-,-,-)$:

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - V(\phi^\dagger \phi)$$

(a)(25 points) Identify the corresponding equations of motion. Consider a special case of $V = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$ separately.

(b)(25 points) Find the energy-momentum tensor for this Lagrangian. Check that this tensor is symmetric and proof it's conservation (using the equations of motion). Write the Hamiltonian for the potential $V = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$.

(c)(25 points) This Lagrangian is symmetric under the transformation $\phi \to \phi \exp(-i\alpha)$ with real α. Find the corresponding Noether current and check (using the equations of motion) that $\partial^\mu j_\mu = 0$.

(d)(25 points) Go back to (a)-(c) and introduce the interaction of ϕ with electromagnetic field via minimal substitution. Gauge invariance guarantees the correct result. Note: if you want to check that $\partial_\mu T^{\mu\nu} = 0$ you have to also include the electromagnetic field term.

(e*)(Advanced level problem for those who are interested - 25 points) In general relativity the action includes the determinant of metric tensor, denoted by g:

$$S = \int \sqrt{-g}\mathcal{L}\, d^n x$$

Energy-momentum tensor arises from the invariance in space-time translations. It appears that one can obtain the expression for this tensor from the following formula:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g_{\mu\nu}}$$

Find $T^{\mu\nu}$ from this formula. Also note that in this case $T^{\mu\nu}$ is automatically symmetric because $g^{\mu\nu}$ is symmetric.

Hint: you will need Jacobi's formula for matrices:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}$$

Literature

1. Quantum Field Theory, Lewis Ryder (mostly chapter 3).