

4.3 Eigenfunktionen von L^2 und L_3

Wir wissen: $[L_i, r] = 0$, also kann L_i keine Ableitungen nach r enthalten

→ transformiere auf Kugelkoordinaten und drücke L_1, L_2, L_3 durch Ableitungen nach θ, φ aus

$$\begin{aligned} x &= r \sin \theta \cos \varphi && \equiv x_1 \\ y &= r \sin \theta \sin \varphi && \equiv x_2 \\ z &= r \cos \theta && \equiv x_3 \end{aligned}$$

Dann ist

$$\begin{aligned} \vec{x} &= r \hat{e}_r \\ \vec{\nabla} &= \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \end{aligned}$$

$$\begin{aligned} \vec{L} &= \frac{\hbar}{i} \vec{x} \times \vec{\nabla} \\ &= \frac{\hbar}{i} r \hat{e}_r \times \left\{ \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right\} \end{aligned}$$

$\{ \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi \}$ bilden ein Rechtssystem

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\varphi, \quad \hat{e}_\theta \times \hat{e}_\varphi = \hat{e}_r, \quad \hat{e}_\varphi \times \hat{e}_r = \hat{e}_\theta$$

$$\Rightarrow \vec{L} = \frac{\hbar}{i} \left\{ \hat{e}_\varphi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right\}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi} \frac{\partial}{\partial z}$$

$$\left(\equiv \sum_{i=1}^3 \frac{\partial x_i}{\partial \varphi} \frac{\partial}{\partial x_i} \right) \quad L_3$$

$$= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \frac{i}{\hbar} (x_1 p_2 - x_2 p_1) = \frac{i}{\hbar} L_3 = i l_3$$

⇒ $L_3 = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}, \quad l_3 = -i \frac{\partial}{\partial \varphi}$

$$\frac{\partial}{\partial \theta} = \sum_{i=1}^3 \frac{\partial x_i}{\partial \theta} \frac{\partial}{\partial x_i}$$

$$= \cos \varphi \left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) - \sin \varphi \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right)$$

$$+ \cos \varphi x_1 \frac{\partial}{\partial x_3} + \sin \varphi x_2 \frac{\partial}{\partial x_3} - r \sin \theta \frac{\partial}{\partial x_3}$$

= 0 after inserting $x_{1,2}(r, \theta, \varphi)$

$= i (\cos \varphi l_2 - \sin \varphi l_1)$ [1]

Anforderung: $\vec{x} \cdot \vec{l} = \vec{x} \cdot (\vec{x} \times \vec{p}) = 0$

$$x_1 l_1 + x_2 l_2 + x_3 l_3 = 0$$

$$\Rightarrow r \sin \theta \cos \varphi l_1 + r \sin \theta \sin \varphi l_2 = -r \cos \theta l_3 \quad \left| \frac{1}{r \cos \theta} \right.$$

$\tan \theta (\cos \varphi l_1 + \sin \varphi l_2) = -l_3$ [2]

$$-\sin\varphi l_1 + \cos\varphi l_2 = -i \frac{\partial}{\partial\theta} \quad [1]$$

$$\cos\varphi l_1 + \sin\varphi l_2 = \cot\theta i \frac{\partial}{\partial\varphi} \quad [2]$$

$$-\sin\varphi [1] + \cos\varphi [2]: \quad l_1 = i \left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} \right)$$

$$\cos\varphi [1] + \sin\varphi [2]: \quad l_2 = i \left(-\cos\varphi \frac{\partial}{\partial\theta} + \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \right)$$

$$l_3 = -i \frac{\partial}{\partial\varphi}$$

Jetzt: Betrachte $\vec{l}^2 = l_+ l_- + l_3^2 - l_3$ und drücke $l_+ l_-$ und \vec{l}^2 durch Ableitungen nach θ, φ aus.

$$l_+ l_- = (l_1 + i l_2)(l_1 - i l_2)$$

$$l_1 + i l_2 = (i \sin\varphi + \cos\varphi) \frac{\partial}{\partial\theta} + (\cos\varphi + i \sin\varphi) i \cot\theta \frac{\partial}{\partial\varphi}$$

$$= e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$l_1 - i l_2 = e^{-i\varphi} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$\Rightarrow \underline{l_+ l_-} = e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) e^{-i\varphi} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$= \dots = \underline{-\frac{\partial^2}{\partial\theta^2} - i \frac{\partial}{\partial\varphi} - \cot\theta \frac{\partial}{\partial\theta} - \cot^2\theta \frac{\partial^2}{\partial\varphi^2}}$$

Daher:

$$\begin{aligned}
 \vec{L}^2 &= L_+ L_- + L_3^2 - L_3 = L_+ L_- - \frac{\partial^2}{\partial \varphi^2} + i \frac{\partial}{\partial \varphi} \\
 &= - \frac{\partial^2}{\partial \theta^2} - \underbrace{(1 + \cot^2 \theta)}_{1/\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} - \cot \theta \frac{\partial}{\partial \theta} \\
 &= - \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right\} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\sin \theta} \frac{\partial \sin \theta}{\partial \theta} \\
 &= - \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta} \right) + \left(\frac{\partial \sin \theta}{\partial \theta} \right) \frac{\partial}{\partial \theta} \right] \right\} \\
 &= - \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right\}
 \end{aligned}$$

Vergleichen: Laplace - Operator in Kugelkoordinaten

$$\begin{aligned}
 \Delta &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right\} \\
 &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \underbrace{\Delta_{\theta, \varphi}}_{\text{Winkelanteil}}
 \end{aligned}$$

Vergleich mit \vec{L}^2 ergibt

$$\vec{L}^2 = -\Delta_{\theta, \varphi} \quad ; \quad \vec{L}^2 = -\hbar^2 \Delta_{\theta, \varphi}$$

Nach diesen länglichen Überlegungen bestimmen wir nun die ... \rightarrow

Eigenfunktionen von \vec{l}^2, l_3

Ortsdarstellung: $|l, m\rangle = Y_{lm}(\theta, \varphi)$

(denn wegen $[L_i, r] = 0$ können L_i keine Ableitungen nach r enthalten)

Wegen $l_3 |l, m\rangle = -i \frac{\partial}{\partial \varphi} Y_{lm}(\theta, \varphi) = m Y_{lm}(\theta, \varphi)$

man $Y_{lm}(\theta, \varphi)$ proportional zu $e^{im\varphi}$ sein:

$$Y_{lm}(\theta, \varphi) = g_{lm}(\theta) e^{im\varphi}$$

$$\begin{aligned} -i \frac{\partial}{\partial \varphi} Y_{lm}(\theta, \varphi) &= g_{lm}(\theta) (-i) im e^{im\varphi} = m g_{lm}(\theta) e^{im\varphi} \\ &= m Y_{lm}(\theta, \varphi) \quad \checkmark \end{aligned}$$

Berechne man

$$l_+ |l, l\rangle = 0$$

$$\Leftrightarrow \underbrace{e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right)}_{l_+} \underbrace{g_{ll}(\theta) e^{il\varphi}}_{Y_{ll}(\theta, \varphi)} = 0$$

$\Gamma \frac{\partial}{\partial \varphi} e^{i\varphi} = i e^{i\varphi}$

$$e^{i(l+1)\varphi} \left(\frac{\partial}{\partial \theta} - l \cot \theta \right) g_{ll}(\theta) = 0$$

$$\Rightarrow \frac{d g_{ll}(\theta)}{d\theta} = l \cot \theta g_{ll}(\theta)$$

$$\Rightarrow \int d(\ln g_{ll}) = l \int \cot \theta \, d\theta + C_l$$

$$\uparrow \int \cot \theta \, d\theta = \int \frac{\cos \theta}{\sin \theta} \, d\theta = \int \frac{\sin' \theta}{\sin \theta} \, d\theta = \int d(\ln \sin \theta)$$

$$\text{wg. } \int \frac{f'}{f} \, dx = \ln f + c \quad \downarrow$$

$$\Rightarrow \int d(\ln g_{ll}) = l \int d(\ln \sin \theta) + C_l$$

$$\Rightarrow \underline{g_{ll}(\theta) = (\sin \theta)^l \cdot C_l}$$

Also ist die Eigenfunktion für $m = m_{\max} = l$:

$$\underline{Y_{ll}(\theta, \varphi) = C_l (\sin \theta)^l e^{i l \varphi}$$

Bestimme C_l aus Normierungsbedingung:

$$\int d\Omega |Y_{ll}|^2 = |C_l|^2 \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta (\sin \theta)^{2l} \stackrel{!}{=} 1$$

$$\Rightarrow |C_l|^2 \cdot 2\pi \int_0^\pi d\theta (\sin \theta)^{2l+1} = 1$$

$$\Rightarrow |C_l|^2 = \frac{1}{4\pi} \frac{2^{2l+1}}{4^l} \binom{2l}{l}$$

Eigenfunktionen $Y_{lm}(\theta, \varphi)$ für $m < l$ erhält man durch sukzessive Anwendung des **Lesteroperators L_-** :

$$L_- g_{ll}(\theta) e^{i l \varphi} = e^{-i \varphi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) g_{ll}(\theta) e^{i l \varphi}$$

$$= - \left(\frac{\partial}{\partial \theta} + l \cot \theta \right) g_{ll}(\theta) e^{i(l-1)\varphi}$$

$$\begin{aligned} \frac{\partial}{\partial \theta} (\sin \theta)^l &= \\ l (\sin \theta)^{l-1} \cos \theta &= \\ = l (\sin \theta)^l \cot \theta & \end{aligned}$$

$$\circ = - (\sin \theta)^{-l} \frac{\partial}{\partial \theta} (\sin \theta)^l g_{ll}(\theta) e^{i(l-1)\varphi}$$

$$(\sin \theta)^{-l} \frac{\partial}{\partial \theta} (\sin \theta)^l = l \cot \theta$$

$$= \underbrace{(\sin \theta)^{-(l-1)} \frac{d}{d \cos \theta} (\sin \theta)^l g_{ll}(\theta)}_{g_{l, l-1}(\theta)} e^{i(l-1)\varphi}$$

\uparrow
 $m=l-1$
 \checkmark

$$\begin{aligned} \frac{d}{d(\cos \theta)} &= \frac{1}{-\sin \theta} \frac{d}{d\theta} \\ \Rightarrow \frac{d}{d\theta} &= -\sin \theta \frac{d}{d \cos \theta} \end{aligned}$$

Nochmalige Anwendung von L_- :

$$\circ L_- g_{l, l-1}(\theta) e^{i(l-1)\varphi}$$

$$= (\sin \theta)^{-(l-2)} \frac{d}{d \cos \theta} (\sin \theta)^{l-1} g_{l, l-1}(\theta) e^{i(l-2)\varphi}$$

$$= (\sin \theta)^{-(l-2)} \frac{d}{d \cos \theta} \underbrace{(\sin \theta)^{l-1}}_1 \underbrace{(\sin \theta)^{-(l-1)}}_{\text{red}} \frac{d}{d \cos \theta} (\sin \theta)^l g_{ll}(\theta) e^{i(l-2)\varphi}$$

$$= (\sin \theta)^{-(l-2)} \left(\frac{d}{d \cos \theta} \right)^2 (\sin \theta)^l g_{ll}(\theta) e^{i(l-2)\varphi}$$

Nach $(l-m)$ -maliger Anwendung: (wie beim harm. Osz.)

$$\begin{aligned}
 & (-1)^{l-m} g_{ll}(\theta) e^{im\varphi} \\
 &= (\sin\theta)^{-(l-l+m)} \left(\frac{d}{d\cos\theta}\right)^{l-m} (\sin\theta)^l g_{ll}(\theta) e^{i(l-l+m)\varphi} \\
 &= (\sin\theta)^{-m} \left(\frac{d}{d\cos\theta}\right)^{l-m} (\sin\theta)^l \underbrace{g_{ll}(\theta)}_{c_l (\sin\theta)^l} e^{im\varphi} \\
 &= c_l (\sin\theta)^{-m} \left(\frac{d}{d\cos\theta}\right)^{l-m} (\sin\theta)^{2l} e^{im\varphi} \propto Y_{lm}(\theta, \varphi)
 \end{aligned}$$

Mit der konventionellen Normierung findet man

$$\begin{aligned}
 Y_{lm}(\theta, \varphi) &= \frac{(-1)^l}{2^l l!} \left(\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!} \right)^{1/2} (\sin\theta)^{-m} \\
 &\quad \times \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l} e^{im\varphi}
 \end{aligned}$$

$Y_{lm}(\theta, \varphi)$ heißen Kugelflächenfunktionen
 (Kugelfunktionen, "spherical harmonics")

explizite Form:

4.3 Eigenfunktionen von \vec{L}^2 und L_3

Gradient in krummlinigen Koordinaten:

$$\vec{x} = x \hat{e}_x + y \hat{e}_y + z \hat{e}_z \equiv x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = q_1 \hat{\eta}_1 + q_2 \hat{\eta}_2 + q_3 \hat{\eta}_3$$

$\hat{\eta}_i$: Basisvektoren in krummlinigen Koordinaten

Maßstabsfaktoren:

$$h_i := \left| \frac{\partial \vec{x}}{\partial q_i} \right|, \quad \hat{\eta}_i = \left(\frac{\partial \vec{x}}{\partial q_i} \right) \frac{1}{h_i}$$

Gradient:

$$\vec{\nabla} = \sum_{i=1}^3 \hat{\eta}_i \frac{1}{h_i} \frac{\partial}{\partial q_i}$$

Kugelkoordinaten: $x \equiv x_1 = r \sin \theta \cos \varphi$, $y \equiv x_2 = r \sin \theta \sin \varphi$, $z \equiv x_3 = r \cos \theta$

$$h_r = \left| \frac{\partial \vec{x}}{\partial r} \right| = 1, \quad h_\theta = \left| \frac{\partial \vec{x}}{\partial \theta} \right| = r, \quad h_\varphi = \left| \frac{\partial \vec{x}}{\partial \varphi} \right| = r \sin \theta \quad \vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = r \hat{e}_r$$

$$\Rightarrow \vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$ bilden ein Rechtssystem:

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\varphi, \quad \hat{e}_\theta \times \hat{e}_\varphi = \hat{e}_r, \quad \hat{e}_\varphi \times \hat{e}_r = \hat{e}_\theta$$

Kugelkoordinaten: $x \equiv x_1 = r \sin \theta \cos \varphi, \quad y \equiv x_2 = r \sin \theta \sin \varphi, \quad z \equiv x_3 = r \cos \theta$

$$\frac{\partial}{\partial \varphi} = \sum_{i=1}^3 \frac{\partial x_i}{\partial \varphi} \frac{\partial}{\partial x_i} \quad \text{mit} \quad \frac{\partial x_1}{\partial \varphi} = -x_2, \quad \frac{\partial x_2}{\partial \varphi} = x_1, \quad \frac{\partial x_3}{\partial \varphi} = 0,$$

$$\frac{\partial}{\partial \theta} = \sum_{i=1}^3 \frac{\partial x_i}{\partial \theta} \frac{\partial}{\partial x_i} \quad \text{mit} \quad \frac{\partial x_1}{\partial \theta} = x_3 \cos \varphi, \quad \frac{\partial x_2}{\partial \theta} = x_3 \sin \varphi, \quad \frac{\partial x_3}{\partial \theta} = r \sin \theta,$$

$$\Rightarrow \frac{\partial}{\partial \varphi} = i l_3, \quad \frac{\partial}{\partial \theta} = i (\cos \varphi l_2 - \sin \varphi l_1) \quad \Rightarrow \quad -\sin \varphi l_1 + \cos \varphi l_2 = -i \frac{\partial}{\partial \theta} \quad [1]$$

$$x_1 l_1 + x_2 l_2 + x_3 l_3 \equiv \vec{x} \cdot \vec{l} = \vec{x} \cdot (\vec{x} \times \vec{p}) = 0$$

$$\Rightarrow \tan \theta (\cos \varphi l_1 + \sin \varphi l_2) = -l_3 \quad \Rightarrow \quad \cos \varphi l_1 + \sin \varphi l_2 = i \cot \theta \frac{\partial}{\partial \varphi} \quad [2]$$

$$-\sin \varphi \ell_1 + \cos \varphi \ell_2 = -i \frac{\partial}{\partial \theta} \quad [1]$$

$$\cos \varphi \ell_1 + \sin \varphi \ell_2 = i \cot \theta \frac{\partial}{\partial \varphi} \quad [2]$$

$$-\sin \varphi \text{ "[1]"} + \cos \varphi \text{ "[2]"} : \quad \ell_1 = i \left(\sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right)$$

$$\cos \varphi \text{ "[1]"} + \sin \varphi \text{ "[2]"} : \quad \ell_2 = i \left(-\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right) \quad \ell_3 = -i \frac{\partial}{\partial \varphi}$$

Drücke $\vec{\ell}^2 = \ell_+ \ell_- + \ell_3^2 - \ell_3$, $\ell_+ \ell_- = (\ell_1 + i\ell_2)(\ell_1 - i\ell_2)$ durch Ableitungen nach θ und φ aus:

$$\ell_1 + i\ell_2 = e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right), \quad \ell_1 - i\ell_2 = e^{-i\varphi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right)$$

$$\Rightarrow \ell_+ \ell_- = e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) e^{-i\varphi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right)$$

$$= \dots = -\frac{\partial^2}{\partial \theta^2} - i \frac{\partial}{\partial \varphi} - \cot \theta \frac{\partial}{\partial \theta} - \cot^2 \theta \frac{\partial^2}{\partial \varphi^2}$$

$$\ell_3^2 - \ell_3 = -\frac{\partial^2}{\partial \varphi^2} + i \frac{\partial}{\partial \varphi}$$

$$\begin{aligned}
\vec{\ell}^2 &= \ell_+ \ell_- + \ell_3^2 - \ell_3 = -\frac{\partial^2}{\partial \theta^2} - i \frac{\partial}{\partial \varphi} - \cot \theta \frac{\partial}{\partial \theta} - \cot^2 \theta \frac{\partial^2}{\partial \varphi^2} - \frac{\partial^2}{\partial \varphi^2} + i \frac{\partial}{\partial \varphi} \\
&= -\frac{\partial^2}{\partial \theta^2} - (1 + \cot^2 \theta) \frac{\partial^2}{\partial \varphi^2} - \cot \theta \frac{\partial}{\partial \theta} = -\left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right\} \\
&= -\left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta} \right) + \left(\frac{\partial \sin \theta}{\partial \theta} \right) \frac{\partial}{\partial \theta} \right] \right\} \\
&= -\left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right\}
\end{aligned}$$

Laplace-Operator in Kugelkoordinaten:

$$\Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left\{ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right\} = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{\theta, \varphi}$$