

3) DIRAC PARTICLE IN CENTRAL POTENTIAL

• STATIONARY STATES IN CENTRAL POTENTIAL $V(r)$

e.g. H-ATOM $V(r) = -\frac{\alpha}{r} \frac{\hbar c}{}$

$$\alpha = \frac{1}{137} = \frac{e^2}{4\pi\epsilon_0 \hbar c}$$

$$\left(c \vec{\alpha} \cdot \hat{p} + \beta m_0 c^2 + V(r) \right) \Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

$$\hat{p} = -i\hbar \vec{\nabla}$$

↓ FOR STATIONARY SOLUTION $\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar} E t} \psi(\vec{r})$
 (V DOES NOT DEPEND ON t)

$$\left(c \vec{\alpha} \cdot \hat{p} + \beta m_0 c^2 + V(r) \right) \psi(\vec{r}) = E \psi(\vec{r})$$

$$\underline{\underline{H_D \psi(\vec{r}) = E \psi(\vec{r})}}$$

DIRAC HAMILTONIAN

$$H_D = c \vec{\alpha} \cdot \hat{p} + \beta m_0 c^2 + V(r)$$

$$\vec{\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• ORBITAL ANGULAR MOMENTUM

$$\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$$

↳ FOR SCHRÖDINGER EQ. (SPINLESS PARTICLE IN CENTRAL POTENTIAL)

↓
 \vec{L}^2, L_z COMMUTE WITH H_S
 ↳ SCHRÖDINGER

↓
 l^2, l_z ARE GOOD QUANTUM NUMBERS FOR SCHRÖDINGER THEORY

↳ IN DIRAC THEORY

$$[H_D, L_i]$$

$$= \epsilon_{ijk} [H_D, r_j \hat{p}_k]$$

$$= \epsilon_{ijk} [c \alpha_\ell \hat{p}_\ell, r_j \hat{p}_k]$$

$$= c \epsilon_{ijk} \alpha_\ell \underbrace{[\hat{p}_\ell, r_j]}_{-i\hbar \delta_{j\ell}} \hat{p}_k$$

$$= -i\hbar c \epsilon_{ijk} \alpha_j \hat{p}_k$$

$$= -i\hbar c (\vec{\alpha} \times \hat{\vec{p}})_i \neq 0$$

$$\Rightarrow [V(r), L_i] = 0$$

$$\text{AS } [\hat{p}_\ell, \hat{p}_k] = 0$$

∴ \vec{L} DOES NOT COMMUTE WITH H_D

↳ l_z IS NOT GOOD QUANTUM NUMBER

• SPIN (INTRINSIC ANGULAR MOMENTUM)

$$\vec{S} = \frac{\hbar}{2} \vec{\Sigma} = \frac{\hbar}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$[H_D, S_i] = \frac{\hbar}{2} [c \vec{\alpha} \cdot \hat{p}, \vec{\Sigma}]$$

$$= \frac{\hbar c}{2} \left\{ \begin{pmatrix} 0 & \vec{\sigma} \cdot \hat{p} \\ \vec{\sigma} \cdot \hat{p} & 0 \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \cdot \hat{p} \\ \vec{\sigma} \cdot \hat{p} & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar c}{2} \begin{pmatrix} 0 & [\vec{\sigma} \cdot \hat{p}, \sigma_i] \\ [\vec{\sigma} \cdot \hat{p}, \sigma_i] & 0 \end{pmatrix}$$

$$\downarrow \quad [\sigma_k, \sigma_l] = 2i \epsilon_{klm} \sigma_m$$

$$= \frac{\hbar c}{2} \begin{pmatrix} 0 & 2i \epsilon_{kij} \hat{p}_k \sigma_j \\ 2i \epsilon_{kij} \hat{p}_k \sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c \epsilon_{ijk} \hat{p}_k \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c \epsilon_{ijk} \hat{p}_k \alpha_j$$

$$= i\hbar c (\vec{\alpha} \times \hat{p})_i \neq 0$$

$\therefore \vec{S}$ DOES NOT COMMUTE WITH H_D

$\hookrightarrow S_z$ IS NOT GOOD QUANTUM NUMBER.

• TOTAL ANGULAR MOMENTUM

$$\vec{J} = \vec{L} + \vec{S}$$

$$[H_D, J_i] = [H_D, L_i] + [H_D, S_i]$$

$$= 0$$

ONLY TOTAL ANGULAR MOMENTUM IS CONSERVED BY DIRAC EQ. (FOR PARTICLE IN CENTRAL POTENTIAL)

$$[H_D, J^2] = 0$$

$$[H_D, J_z] = 0$$

$$[J^2, J_i] = 0$$

IN DIRAC EQ. THERE IS A COUPLING BETWEEN SPIN \vec{S} AND ORBITAL ANGULAR MOMENTUM \vec{L} \Rightarrow SPIN-ORBIT INTERACTION

BUT $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

$\therefore H_D, J^2, J_z$ COMMUTE SIMULTANEOUSLY

SOLUTIONS OF H_D HAVE GOOD QUANTUM NUMBERS j, j_z

$$\begin{cases} J^2 \psi = \hbar^2 j(j+1) \psi \\ J_z \psi = \hbar j_z \psi \end{cases}$$

$\Rightarrow j = l \pm \frac{1}{2}$

FOR GIVEN VALUE OF j , 2 SOLUTIONS \hookrightarrow WE NEED ANOTHER CONSERVED QUANTITY (GOOD QUANTUM NUMBER) TO DISTINGUISH BOTH CASES.

• CONSIDER

OPERATOR

$$K \equiv \beta (\bar{\Sigma} \cdot \bar{L} + \hbar)$$

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WILL SHOW THAT $[H_D, K] = 0$

↳ $\bar{\Sigma} \cdot \bar{L}$ SPIN-ORBIT OPERATOR

CAN BE WRITTEN EQUIVALENTLY AS $\bar{\Sigma} \cdot \bar{L} = \bar{\Sigma} \cdot (\bar{J} - \frac{\hbar}{2} \bar{\Sigma})$
 $= \bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2} \bar{\Sigma}^2$
 $= \bar{\Sigma} \cdot \bar{J} - \frac{3\hbar}{2} \mathbb{1}$ ($\bar{\Sigma}^2 = 3 \mathbb{1}$)

$$\begin{aligned} K &= \beta (\bar{\Sigma} \cdot \bar{L} + \hbar) \\ &= \beta (\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}) \end{aligned}$$

↳ PROOF $[H_D, K] = 0$

$$[H_D, K]$$

$$= [H_D, \beta] (\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}) + \beta [H_D, \bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}]$$

$$= c [\bar{\alpha} \cdot \hat{P}, \beta] (\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}) + \beta [H_D, \bar{\Sigma} \cdot \bar{J}]$$

$$\downarrow \text{AS } \alpha_i \beta = -\beta \alpha_i$$

$$\downarrow \text{AS } [H_D, J_i] = 0$$

$$= -2c\beta (\bar{\alpha} \cdot \hat{P}) (\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}) + \beta [H_D, \Sigma_i] J_i$$

$$= -2c\beta \begin{pmatrix} 0 & \bar{\sigma} \cdot \hat{P} \\ +\bar{\sigma} \cdot \hat{P} & 0 \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} J_i \quad \downarrow [H_D, \Sigma_i] = 2ic(\bar{\alpha} \times \hat{P})_i$$

$$+ \hbar c \beta (\bar{\alpha} \cdot \hat{P}) + 2ic \beta (\bar{\alpha} \times \hat{P}) \cdot \bar{J}$$

$$\downarrow (\vec{\sigma} \cdot \hat{P})(\vec{\sigma} \cdot \vec{J}) = \hat{P} \cdot \vec{J} + i \vec{\sigma} \cdot (\hat{P} \times \vec{J})$$

$$(\vec{\sigma} \cdot \hat{P}) \sigma_i = \hat{P}_i + i \epsilon_{kij} \hat{P}_k \sigma_j$$

$$[H_D, K]$$

$$= -2c\beta \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \hat{P} \cdot \vec{J} - 2c\beta i \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_j \hat{P}_k \\ \sigma_j \hat{P}_k & 0 \end{pmatrix} J_i$$

$$+ c\hbar\beta \hat{P} \cdot \begin{pmatrix} 0 & \mathbb{1} \\ \vec{\sigma} & 0 \end{pmatrix} + 2c\beta i \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_j \hat{P}_k \\ \sigma_j \hat{P}_k & 0 \end{pmatrix} J_i$$

$$= -2c\beta \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \hat{P} \cdot \vec{J}$$

$$+ 2c\beta \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \hat{P} \cdot \begin{pmatrix} \frac{\hbar}{2} \mathbb{1} & 0 \\ 0 & \frac{\hbar}{2} \mathbb{1} \end{pmatrix}$$

$$= -2c\beta \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}_{\gamma_5} \cdot \hat{P} \cdot \underbrace{\left(\vec{J} - \frac{\hbar}{2} \vec{\Sigma} \right)}_{\vec{L}}$$

$$\downarrow \hat{P} \cdot \vec{L} = 0$$

$$\stackrel{\nabla}{=} 0$$

↳ PROOF $[K, J_i] = 0$

$$\begin{aligned}
 & [K, J_i] \\
 &= \left[\beta \left(\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2} \right), J_i \right] \\
 &= [\beta, J_i] \left(\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2} \right) + \beta \left[\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2}, J_i \right] \\
 &= \frac{\hbar}{2} \underbrace{[\beta, \Sigma_i]}_0 \left(\bar{\Sigma} \cdot \bar{J} - \frac{\hbar}{2} \right) + \beta \left[\bar{\Sigma} \cdot \bar{J}, J_i \right] \\
 &= \beta \sum_j \underbrace{[J_j, J_i]}_{i\hbar \epsilon_{jik} J_k} + \beta \underbrace{[\Sigma_j, J_i]}_{\frac{\hbar}{2} [\Sigma_j, \Sigma_i]} J_j \\
 & \hspace{15em} = \frac{\hbar}{2} 2i \epsilon_{jik} \Sigma_k \\
 &= i\hbar \epsilon_{jck} \beta \sum_j J_k + i\hbar \epsilon_{jik} \beta \sum_k J_j \\
 &= 0
 \end{aligned}$$

∴ H_D, J^2, J_z, K FORM A SET OF SIMULTANEOUSLY COMMUTING OPERATORS

⇓
HAVE COMMON EIGENSTATES

• EIGENVALUES OF $K = \beta(\bar{\Sigma} \cdot \bar{L} + \hbar)$

↳ DEFINE EIGENVALUE OF K AS $-\hbar K$

$$\underline{\underline{K\psi = -\hbar K\psi}}$$

↳ DETERMINE K^2

$$K^2 = \beta(\bar{\Sigma} \cdot \bar{L} + \hbar) \beta(\bar{\Sigma} \cdot \bar{L} + \hbar)$$

$$\downarrow \beta \Sigma_i = \Sigma_i \beta$$

$$= (\Sigma_i L_i) (\Sigma_j L_j) + 2\hbar (\bar{\Sigma} \cdot \bar{L}) + \hbar^2$$

$$\downarrow \Sigma_i \Sigma_j = \delta_{ij} + i \epsilon_{ijk} \Sigma_k$$

$$= L^2 + i \epsilon_{ijk} L_i L_j \Sigma_k + 2\hbar (\bar{\Sigma} \cdot \bar{L}) + \hbar^2$$

$$= L^2 + \frac{i}{2} \epsilon_{ijk} \underbrace{(L_i L_j - L_j L_i)}_{i\hbar \epsilon_{ijl} L_l} \Sigma_k + 2\hbar (\bar{\Sigma} \cdot \bar{L}) + \hbar^2$$

$$\downarrow \epsilon_{ijk} \epsilon_{ijl} = 2 \delta_{kl}$$

$$= L^2 - \hbar (\bar{\Sigma} \cdot \bar{L}) + 2\hbar (\bar{\Sigma} \cdot \bar{L}) + \hbar^2$$

$$= L^2 + 2(\bar{S} \cdot \bar{L}) + \hbar^2$$

$$= (\bar{L} + \bar{S})^2 - \bar{S}^2 + \hbar^2$$

$$= J^2 + \frac{\hbar^2}{4} \mathbb{1}$$

$$\underline{\underline{K^2 = J^2 + \frac{\hbar^2}{4} \mathbb{1}}}$$

$$\bar{S}^2 = \frac{\hbar^2}{4} \Sigma^2 = \frac{3\hbar^2}{4} \mathbb{1}$$

FOR EIGENSTATE OF \mathbf{J}

$$\mathbf{J}^2 \Psi = \hbar^2 j(j+1) \Psi$$

$$\downarrow$$

$$K^2 \Psi = \left(\mathbf{J}^2 + \frac{\hbar^2}{4} \right) \Psi$$

$$= \hbar^2 \left(j(j+1) + \frac{1}{4} \right) \Psi$$

$$= \hbar^2 \left(j + \frac{1}{2} \right)^2 \Psi$$

$$= \hbar^2 K^2 \Psi$$

$$\boxed{|K| = j + \frac{1}{2}}$$

$$\underline{\underline{K = \pm \left(j + \frac{1}{2} \right)}}$$

↳ SIGN OF K

$$\underline{\underline{K \Psi = \mp \hbar \left(j + \frac{1}{2} \right) \Psi}}$$

$$\beta \begin{pmatrix} \vec{\sigma} \cdot \vec{L} + \hbar & 0 \\ 0 & \vec{\sigma} \cdot \vec{L} + \hbar \end{pmatrix} \Psi = \mp \hbar \left(j + \frac{1}{2} \right) \Psi$$

$$\downarrow \Psi = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$$

WITH Ψ_A, Ψ_B : 2x1 PAULI SPINORS

$$\begin{pmatrix} \vec{\sigma} \cdot \vec{L} + \hbar & 0 \\ 0 & -\vec{\sigma} \cdot \vec{L} - \hbar \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \mp \hbar \left(j + \frac{1}{2} \right) \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$$

$$\begin{cases} (\vec{\sigma} \cdot \vec{L}) \psi_A = +\hbar \left(j + \frac{1}{2} \pm 1 \right) \psi_A \\ -(\vec{\sigma} \cdot \vec{L}) \psi_B = +\hbar \left(j + \frac{1}{2} \mp 1 \right) \psi_B \end{cases}$$

$$\text{FOR } \underline{K = + \left(j + \frac{1}{2} \right)} \quad \begin{cases} (\vec{\sigma} \cdot \vec{L}) \psi_A = -\hbar \left(j + \frac{3}{2} \right) \psi_A \\ (\vec{\sigma} \cdot \vec{L}) \psi_B = \hbar \left(j - \frac{1}{2} \right) \psi_B \end{cases}$$

$$\underline{K = - \left(j + \frac{1}{2} \right)} \quad \begin{cases} (\vec{\sigma} \cdot \vec{L}) \psi_A = \hbar \left(j - \frac{1}{2} \right) \psi_A \\ (\vec{\sigma} \cdot \vec{L}) \psi_B = -\hbar \left(j + \frac{3}{2} \right) \psi_B \end{cases}$$

ψ_A, ψ_B ARE EIGENSTATES OF $\vec{\sigma} \cdot \vec{L}$
BUT WITH DIFFERENT EIGENVALUES

$$\hookrightarrow L^2 = J^2 - \hbar \vec{\sigma} \cdot \vec{L} - \frac{3}{4} \hbar^2$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \left(j(j+1) - \frac{3}{4} \right) \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \hbar \begin{pmatrix} \vec{\sigma} \cdot \vec{L} \psi_A \\ \vec{\sigma} \cdot \vec{L} \psi_B \end{pmatrix}$$

$$\text{FOR } \underline{K = + \left(j + \frac{1}{2} \right)}$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \left[j(j+1) - \frac{3}{4} \right] \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} + \hbar^2 \begin{pmatrix} \left(j + \frac{3}{2} \right) \psi_A \\ - \left(j - \frac{1}{2} \right) \psi_B \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} \left(j^2 + 2j + \frac{3}{4} \right) \psi_A \\ \left(j^2 - \frac{1}{4} \right) \psi_B \end{pmatrix}$$

DENOTE $\underline{\underline{l_{\pm} \equiv j \pm \frac{1}{2}}}$

$$l_+(l_+ + 1) = j^2 + 2j + \frac{3}{4}$$

$$l_-(l_- + 1) = j^2 - \frac{1}{4}$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} l_+(l_+ + 1) \psi_A \\ l_-(l_- + 1) \psi_B \end{pmatrix}$$

ψ_A, ψ_B ARE EIGENSTATES OF L^2
BUT WITH DIFFERENT EIGENVALUES.

↳ DIRAC SPINOR NOT EIGENSTATE OF L^2

FOR $K = -(j + \frac{1}{2})$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} l_-(l_- + 1) \psi_A \\ l_+(l_+ + 1) \psi_B \end{pmatrix}$$

$$\begin{array}{l} \circ \circ \\ \parallel \\ K = +(j + \frac{1}{2}) > 0 \Rightarrow l_A = l_+, l_B = l_- \\ K = -(j + \frac{1}{2}) < 0 \Rightarrow l_A = l_-, l_B = l_+ \end{array}$$

NOTE SIGN OF K INDICATES WHETHER \bar{L} & $\bar{\sigma}$ ARE // OR ANTI-//

(OPPOSITE FOR UPPER COMPONENT ψ_A & LOWER COMPONENT ψ_B)

$$\begin{cases} (\bar{\sigma} \cdot \bar{L}) \psi_A = -\hbar (K + 1) \psi_A \\ (\bar{\sigma} \cdot \bar{L}) \psi_B = +\hbar (K - 1) \psi_B \end{cases}$$

SOLUTION OF DIRAC EQ. IN CENTRAL POTENTIAL.

$\hookrightarrow H_D \psi = E \psi \quad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$

$$\begin{pmatrix} 0 & c \vec{\sigma} \cdot \hat{p} \\ c \vec{\sigma} \cdot \hat{p} & 0 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} (E - V(r) - m_0 c^2) \psi_A \\ (E - V(r) + m_0 c^2) \psi_B \end{pmatrix}$$

$$\begin{cases} c (\vec{\sigma} \cdot \hat{p}) \psi_B = (E - V(r) - m_0 c^2) \psi_A \\ c (\vec{\sigma} \cdot \hat{p}) \psi_A = (E - V(r) + m_0 c^2) \psi_B \end{cases}$$

DENOTE $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r) \varphi_{j l_A m} \\ i f(r) \varphi_{j l_B m} \end{pmatrix}$

ψ HAS GOOD j, m, k

$$\begin{cases} J^2 \psi = \hbar^2 j(j+1) \psi \\ J_z \psi = \hbar m \psi \\ K \psi = -\hbar k \psi \quad \text{WITH } k = \pm(j + \frac{1}{2}) \end{cases}$$

FOR $k > 0 \rightarrow l_A = j + \frac{1}{2}$
 $l_B = j - \frac{1}{2}$

$k < 0 \rightarrow l_A = j - \frac{1}{2}$
 $l_B = j + \frac{1}{2}$

$\varphi_{j l m}$ DEPENDS ONLY ON ANGLES
 \hookrightarrow EIGENSTATES OF J^2, J_z, L^2

PAULI SPINOR
 \downarrow

$$\varphi_{j l m}(\vec{e}_r) = \sum_{m_l} \sum_{m_s} \langle l m_l, \frac{1}{2} m_s | j m \rangle Y_{l m_l}(\vec{e}_r) \chi_{m_s}$$

$$\hookrightarrow \hat{\sigma} \cdot \hat{p} = \underbrace{(\hat{\sigma} \cdot \bar{e}_r)}_1 (\hat{\sigma} \cdot \bar{e}_r) (\hat{\sigma} \cdot \hat{p}) \quad \bar{e}_r = \frac{\bar{r}}{r} \quad \text{IV } 82$$

$$= (\hat{\sigma} \cdot \bar{e}_r) \left\{ \bar{e}_r \cdot \hat{p} + i \hat{\sigma} \cdot (\bar{e}_r \times \hat{p}) \right\}$$

$$\downarrow \quad \bar{e}_r \cdot \hat{p} = -i\hbar \bar{e}_r \cdot \bar{\nabla}$$

$$= -i\hbar \frac{\partial}{\partial r}$$

$$= (\hat{\sigma} \cdot \bar{e}_r) \left\{ -i\hbar \frac{\partial}{\partial r} + \frac{i}{r} \hat{\sigma} \cdot \underbrace{(\bar{r} \times \hat{p})}_{\bar{L}} \right\}$$

$$\underline{\underline{\hat{\sigma} \cdot \hat{p} = (\hat{\sigma} \cdot \bar{e}_r) \left\{ -i\hbar \frac{\partial}{\partial r} + \frac{i}{r} \hat{\sigma} \cdot \bar{L} \right\}}}$$

NOTE $\hat{\sigma} \cdot \bar{e}_r$ & $\hat{\sigma} \cdot \bar{L}$ ONLY ACT ON ANGULAR PARTS OF WAVE FUNCTION.

$$\begin{cases} c(\hat{\sigma} \cdot \hat{p}) \psi_B = (E - V(r) - m_0 c^2) \psi_A \\ c(\hat{\sigma} \cdot \hat{p}) \psi_A = (E - V(r) - m_0 c^2) \psi_B \end{cases}$$

$\hat{\Downarrow}$

$$\left\{ \begin{aligned} & c(\hat{\sigma} \cdot \bar{e}_r) \left\{ \hbar f'(r) - \frac{\hbar f(r)(K-1)}{r} \right\} \psi_{j l_B m} \\ & = (E - V(r) - m_0 c^2) g(r) \psi_{j l_A m} \\ & c(\hat{\sigma} \cdot \bar{e}_r) \left\{ -i\hbar g'(r) - i\hbar \frac{g(r)(K+1)}{r} \right\} \psi_{j l_A m} \\ & = (E - V(r) + m_0 c^2) i f(r) \psi_{j l_B m} \end{aligned} \right.$$

$$\hookrightarrow (\vec{\sigma} \cdot \vec{e}_r) \Psi_{j\ell m}$$

$$\rightsquigarrow [\vec{\sigma} \cdot \vec{e}_r, J_i]$$

$$= \frac{1}{r} [\vec{\sigma} \cdot \vec{r}, (\vec{r} \times \hat{p})_i + \frac{\hbar}{2} \sigma_i]$$

$$= \frac{1}{r} \left\{ \epsilon_{ijk} \sigma_e \underbrace{[r_e, r_j \hat{p}_k]}_{r_j [\hat{p}_e, \hat{p}_k]} + \frac{\hbar}{2} r_e \underbrace{[\sigma_e, \sigma_i]}_{2i \epsilon_{eij} \sigma_j} \right\}$$

$$= \frac{i\hbar}{r} \left\{ \epsilon_{ijk} r_j \sigma_k + \epsilon_{ijl} \sigma_j r_l \right\}$$

$$= 0$$

$\therefore (\vec{\sigma} \cdot \vec{e}_r) \Psi_{j\ell m}$ IS EIGENSTATE OF J^2, J_z
WITH QUANTUM NUMBERS j, m

\rightsquigarrow UNDER SPATIAL INVERSION

$$\vec{r} \rightarrow -\vec{r}$$

$$\vec{\sigma} \rightarrow \vec{\sigma}$$

$$(\vec{\sigma} \cdot \vec{e}_r) \rightarrow -(\vec{\sigma} \cdot \vec{e}_r)$$

TRANSFORMS AS
PSEUDOSCALAR

$$\Psi_{j\ell m} : \text{PARITY } (-1)^\ell$$

$(\vec{\sigma} \cdot \vec{e}_r)$ CHANGES PARITY OF STATE

$$(\vec{\sigma} \cdot \vec{e}_r) \underbrace{\Psi_{j\ell_A m}}_{\text{PARITY } (-1)^{\ell_A}} = C \underbrace{\Psi_{j\ell_B m}}_{(-1)^{\ell_B} = -(-1)^{\ell_A}} \quad \text{AS } \underline{\ell_B = \ell_A \mp 1}$$

$$(\vec{\sigma} \cdot \vec{e}_r)^2 = 1 \Rightarrow C^2 = 1 \Rightarrow C = \pm 1$$

$$\frac{(\vec{\sigma} \cdot \vec{e}_r) \Psi_{j l_A m} = - \Psi_{j l_B m}}{\text{PHASE CONVENTION}}$$

CHECK FOR $l_A = 0$

$$(\vec{\sigma} \cdot \vec{e}_r) = \sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z$$

$$\downarrow \quad S_{\pm} = \frac{\hbar}{2} (\sigma_x \pm i \sigma_y)$$

$$\hbar \sigma_x = S_+ + S_-$$

$$\hbar \sigma_y = -i(S_+ - S_-)$$

$$\hbar (\vec{\sigma} \cdot \vec{e}_r) = \sin \theta e^{-i\phi} S_+ + \sin \theta e^{+i\phi} S_- + 2 \cos \theta S_z$$

$$\downarrow \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_{1,-1} = +\sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$\hbar (\vec{\sigma} \cdot \vec{e}_r) = 2 \sqrt{\frac{2\pi}{3}} Y_{1,-1} S_+ - 2 \sqrt{\frac{2\pi}{3}} Y_{1,1} S_- + 2 \sqrt{\frac{4\pi}{3}} S_z$$

SPECIAL CASE $l_A = 0$, $l_B = 1$, $j = +\frac{1}{2}$, $m = +\frac{1}{2}$

$$\Psi_{\frac{1}{2} 0 \frac{1}{2}}(\vec{e}_r) = Y_{00}(\vec{e}_r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \chi_{+\frac{1}{2}}$$

$$\Psi_{\frac{1}{2} 1 \frac{1}{2}}(\vec{e}_r) = \sum_{m_l m_s} \langle 1 m_l, \frac{1}{2} m_s | \frac{1}{2} + \frac{1}{2} \rangle Y_{1 m_l}(\vec{e}_r) \chi_{m_s}$$

$$\begin{aligned} \Psi_{\frac{1}{2} 1 \frac{1}{2}}(\bar{e}_\kappa) &= \langle 1 0, \frac{1}{2} + \frac{1}{2} | \frac{1}{2} + \frac{1}{2} \rangle Y_{10}(\bar{e}_\kappa) \chi_{+\frac{1}{2}} \\ &+ \langle 1 + 1, \frac{1}{2} - \frac{1}{2} | \frac{1}{2} + \frac{1}{2} \rangle Y_{1+1}(\bar{e}_\kappa) \chi_{-\frac{1}{2}} \\ &= -\sqrt{\frac{1}{3}} Y_{10}(\bar{e}_\kappa) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}} Y_{1+1}(\bar{e}_\kappa) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$${}^{\circ} \hbar (\bar{\sigma} \cdot \bar{e}_\kappa) \Psi_{\frac{1}{2} 0 \frac{1}{2}}(\bar{e}_\kappa)$$

$$= -2 \sqrt{\frac{2\hbar}{3}} \frac{1}{\sqrt{4\pi}} Y_{1,+1}(\bar{e}_\kappa) \underbrace{S_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$+ 2 \sqrt{\frac{\hbar}{3}} \frac{1}{\sqrt{4\pi}} Y_{1,0}(\bar{e}_\kappa) \underbrace{S_z \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= \frac{\hbar}{2} \left\{ -\sqrt{\frac{2}{3}} Y_{1+1}(\bar{e}_\kappa) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{\frac{1}{3}} Y_{10}(\bar{e}_\kappa) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= -\hbar \Psi_{\frac{1}{2} 1 \frac{1}{2}}(\bar{e}_\kappa)$$

$${}^{\circ} \hbar (\bar{\sigma} \cdot \bar{e}_\kappa) \Psi_{\frac{1}{2} 0 \frac{1}{2}}(\bar{e}_\kappa) = -\Psi_{\frac{1}{2} 1 \frac{1}{2}}(\bar{e}_\kappa)$$

↳ COUPLED RADIAL EQUATIONS FOR f, g

$$\begin{cases} -\hbar c \left\{ f' - \frac{f}{r} (k-1) \right\} = (E - V(r) - m_0 c^2) g \\ -\hbar c \left\{ -g' - \frac{g}{r} (k+1) \right\} = (E - V(r) + m_0 c^2) f \end{cases}$$

$$\downarrow \quad g(r) \equiv \frac{G(r)}{r}, \quad f(r) \equiv \frac{F(r)}{r}$$

$$\begin{cases} \frac{\partial F}{\partial r} - k \frac{F}{r} = \frac{1}{\hbar c} (-E + V(r) + m_0 c^2) G \\ \frac{\partial G}{\partial r} + k \frac{G}{r} = \frac{1}{\hbar c} (E - V(r) + m_0 c^2) F \end{cases}$$

↳ TOTAL SOLUTION

$$\Psi = \begin{pmatrix} \frac{G(r)}{r} \varphi_{j l_A m}(\bar{e}_r) \\ i \frac{F(r)}{r} \varphi_{j l_B m}(\bar{e}_r) \end{pmatrix}$$

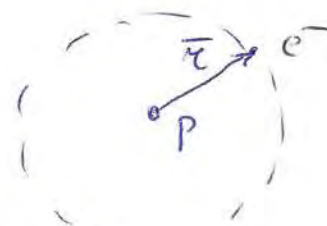
$$\text{FOR } k = + \left(j + \frac{1}{2} \right) \quad l_A = j + \frac{1}{2}, \quad l_B = j - \frac{1}{2}$$

$$k = - \left(j + \frac{1}{2} \right) \quad l_A = j - \frac{1}{2}, \quad l_B = j + \frac{1}{2}$$

• HYDROGEN ATOM IN DIRAC THEORY

↳ BOUND STATES IN

COULOMB POTENTIAL OF e^- ORBITING AROUND PROTON



$$V(r) = -\frac{Z\alpha \hbar c}{r}$$

FOR PROTON $Z = +1$
 He^+ $Z = +2, \dots$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$$

↳ FINE-STRUCTURE CONSTANT

↳ INTRODUCE

$$k_1 \equiv \frac{1}{\hbar c} (E + m_0 c^2)$$

$$k_2 \equiv \frac{1}{\hbar c} (-E + m_0 c^2)$$

FOR BOUND STATES $E < m_0 c^2 \Rightarrow k_2 > 0$

$$\begin{cases} \frac{\partial F}{\partial r} - k \frac{F}{r} = \left(k_2 - \frac{Z\alpha}{r} \right) G \\ \frac{\partial G}{\partial r} + k \frac{G}{r} = \left(k_1 + \frac{Z\alpha}{r} \right) F \end{cases}$$

DIMENSIONLESS VARIABLE

$$\rho \equiv 2\sqrt{k_1 k_2} \quad r = \frac{2}{\hbar c} (m_0^2 c^4 - E^2)^{1/2} r$$

$$\begin{cases} \frac{\partial F}{\partial \rho} - k \frac{F}{\rho} = \left(\frac{1}{2} \sqrt{\frac{k_2}{k_1}} - \frac{Z\alpha}{\rho} \right) G \\ \frac{\partial G}{\partial \rho} + k \frac{G}{\rho} = \left(\frac{1}{2} \sqrt{\frac{k_1}{k_2}} + \frac{Z\alpha}{\rho} \right) F \end{cases}$$

↳ ASYMPTOTIC FORM $\rho \rightarrow \infty$

$$\frac{\partial F}{\partial \rho} \approx \frac{1}{2} \sqrt{\frac{k_2}{k_1}} G \quad \Rightarrow \quad \frac{\partial^2 F}{\partial \rho^2} = \frac{1}{4} F$$

$$\frac{\partial G}{\partial \rho} \approx \frac{1}{2} \sqrt{\frac{k_1}{k_2}} F \quad \Rightarrow \quad \frac{\partial^2 G}{\partial \rho^2} = \frac{1}{4} G$$

$$\therefore F(\rho) \underset{\rho \rightarrow \infty}{\sim} e^{-\rho/2}$$

$$G(\rho) \underset{\rho \rightarrow \infty}{\sim} e^{-\rho/2}$$

↳ SOLUTION IN POWER SERIES FORM

$$F(\rho) = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{\infty} a_m \rho^{m+\delta}$$

$$G(\rho) = \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{\infty} b_m \rho^{m+\delta}$$

BEHAVIOR FOR $\rho \rightarrow 0$

$$F(\rho) \sim \rho^{\delta}$$

$$G(\rho) \sim \rho^{\delta}$$

↳ RECURSION RELATIONS

$$\frac{\partial F}{\partial \rho} = -\frac{1}{2} F + \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{\infty} a_m (m+\delta) \rho^{m+\delta-1}$$

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2} G + \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{\infty} b_m (m+\delta) \rho^{m+\delta-1}$$

$$\hookrightarrow \frac{\partial F}{\partial \rho} - K \frac{F}{\rho} = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{\infty} \left\{ -\frac{1}{2} a_m \rho^{m+\gamma} - K a_m \rho^{m+\gamma-1} + a_m (m+\gamma) \rho^{m+\gamma-1} \right\}$$

↓ IN 10 TERM : INTRODUCE $\underline{a_{-1} = 0}$
 & EXTEND SUM $-1 \rightarrow +\infty$

$$= \sqrt{k_2} e^{-\rho/2} \left\{ - \sum_{m=-1}^{\infty} \frac{1}{2} a_m \rho^{m+\gamma} + \sum_{m=0}^{\infty} a_m \rho^{m+\gamma-1} (-K + m + \gamma) \right\}$$

↓
 $m' = m - 1$

$$= \sqrt{k_2} e^{-\rho/2} \sum_{m=-1}^{\infty} \rho^{m+\gamma} \left\{ -\frac{1}{2} a_m + a_{m+1} (m+1+\gamma-K) \right\}$$

$$\hookrightarrow \left(\frac{1}{2} \sqrt{\frac{k_2}{k_1}} - \frac{Z\alpha}{\rho} \right) G = \sqrt{k_2} e^{-\rho/2} \sum_{m=-1}^{\infty} \left\{ \frac{1}{2} b_m - Z\alpha \frac{k_1}{\sqrt{k_1 k_2}} b_{m+1} \right\} \rho^{m+\delta}$$

WITH $\underline{b_{-1} = 0}$

∴ EQUATE BOTH SIDES

$$\boxed{-\frac{1}{2} a_m + (m+1+\gamma-K) a_{m+1} = \frac{1}{2} b_m - \frac{Z\alpha (E+m_0 c^2)}{\hbar c \sqrt{k_1 k_2}} b_{m+1}}$$

ANALOGOUSLY $a \leftrightarrow b$, $K \leftrightarrow -K$, $Z\alpha \rightarrow -Z\alpha$, $k_1 \leftrightarrow k_2$

$$\boxed{-\frac{1}{2} b_m + (m+1+\gamma+K) b_{m+1} = \frac{1}{2} a_m + \frac{Z\alpha (-E+m_0 c^2)}{\hbar c \sqrt{k_1 k_2}} a_{m+1}}$$

↳ TERM FOR $m = -1$

$$a_{-1} = b_{-1} = 0$$

$$\begin{cases} (\gamma - K) a_0 = - \frac{Z\alpha (E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}} b_0 \\ (\gamma + K) b_0 = + \frac{Z\alpha (-E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}} a_0 \end{cases}$$

⇓

$$\begin{aligned} (\gamma^2 - K^2) &= - (Z\alpha)^2 \frac{m_0^2 c^4 - E^2}{(\hbar c)^2 k_1 k_2} \\ &= - (Z\alpha)^2 \end{aligned}$$

CONDITION TO HAVE A REGULAR SOLUTION AT $r=0$

$$\gamma = \sqrt{K^2 - (Z\alpha)^2}$$

$$\gamma = \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}$$

↳ ADD EQUATIONS $\delta m \rightarrow m-1$

$$\begin{aligned} & (m + \gamma + \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}}) (a_m + b_m) \\ &= (a_{m-1} + b_{m-1}) + \left(K + \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right) (a_m - b_m) \end{aligned}$$

↳ SUBTRACT EQUATIONS $\delta m \rightarrow m-1$

$$\begin{aligned} & (m + \gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}}) (a_m - b_m) \\ \pi &= \left(K - \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right) (a_m + b_m) \\ \circ \circ & \parallel (a_m - b_m) = \frac{\left(K - \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right)}{\left(m + \gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} \right)} (a_m + b_m) \end{aligned}$$

↳ PLUG INTO 1^o EQ.

$$\begin{aligned} & (a_m + b_m) \left[(m + \gamma)^2 - \frac{(Z\alpha)^2 E^2}{(\hbar c)^2 k_1 k_2} - K^2 + \frac{(Z\alpha)^2 m_0^2 c^4}{(\hbar c)^2 k_1 k_2} \right] \\ &= (m + \gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}}) (a_{m-1} + b_{m-1}) \\ & \quad \Downarrow \\ & (a_m + b_m) \left[m^2 + 2m\gamma + \gamma^2 + (Z\alpha)^2 - K^2 \right] \\ &= \left(m + \gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} \right) (a_{m-1} + b_{m-1}) \\ & \quad \Downarrow \quad \gamma^2 = K^2 - (Z\alpha)^2 \end{aligned}$$

$$(a_m + b_m) = \frac{m - \left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right)}{m (2\gamma + m)} (a_{m-1} + b_{m-1})$$

$$= \frac{\left[1 - \left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right) \right] \dots \left[m - \left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right) \right]}{m! (2\gamma + 1) \dots (2\gamma + m)} (a_0 + b_0)$$

$$(a_m - b_m) = \frac{\left(K - \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right)}{\left[m - \left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right) \right]} (a_m + b_m)$$

$$\downarrow (a_0 + b_0) = \frac{\left(\gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} \right)}{\left(K - \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right)} (a_0 - b_0)$$

$$(a_m - b_m) = (-1)^m \frac{\left[\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right] \left[\left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right) - 1 \right] \dots \left[\left(\frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \right) - m + 1 \right]}{m! (2\gamma + 1) \dots (2\gamma + m)} (a_0 - b_0)$$

SERIES CAN BE EXPRESSED THROUGH CONFLUENT HYPERGEOMETRIC FUNCTION

$${}_1F_1(a, b; z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

$$= 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \dots$$

POCHHAMMER SYMBOL $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$

↳ FOR NORMALIZABLE SOLUTIONS

SERIES MUST TERMINATE FOR $m = m' \geq 0$

i.e. $a_{m'+1} = b_{m'+1} = 0$

⇓

$$\underline{a_{m'} = -b_{m'}}$$

$$(a_{m'} - b_{m'}) \left(m' + \gamma - \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} \right)$$

$$= (a_{m'} + b_{m'}) \left(K - \frac{Z\alpha m_0 c^2}{\hbar c \sqrt{k_1 k_2}} \right)$$

$$= 0$$

⇓

$$\boxed{m' = \frac{Z\alpha E}{\hbar c \sqrt{k_1 k_2}} - \gamma \geq 0}$$

INTEGER
 $m' = 0, 1, \dots$

$$\left(\frac{m_0^2 c^4}{E^2} - 1 \right)^{1/2} = \frac{Z\alpha}{(m' + \gamma)}$$

$$E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{(m' + \gamma)^2} \right]^{1/2}}$$

$$\Downarrow \quad \gamma = \left[\left(j + \frac{1}{2} \right)^2 - (Z\alpha)^2 \right]^{1/2}$$

$$\boxed{E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{\left(m' + \sqrt{\left(j + \frac{1}{2} \right)^2 - (Z\alpha)^2} \right)^2} \right]^{1/2}}}$$

$m' = 0, 1, \dots$

L → PERTURBATIVE EXPANSION IN $(Z\alpha)$

$$(Z\alpha) \ll 1$$

FOR $Z=1$, $\alpha = \frac{1}{137}$: $Z\alpha \approx 10^{-2} \ll 1$

$$E \approx m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{\left(n' + j + \frac{1}{2} - \frac{Z^2 \alpha^2}{2(j + \frac{1}{2})} \right)^2} \right]^{-1/2}$$

$$\approx m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n' + j + \frac{1}{2})^2} \left(1 - \frac{(Z\alpha)^2}{(j + \frac{1}{2})(n' + j + \frac{1}{2})} \right) \right]^{-1/2}$$

$$\approx m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n' + j + \frac{1}{2})^2} + \frac{(Z\alpha)^4}{(j + \frac{1}{2})(n' + j + \frac{1}{2})^3} \right]^{-1/2}$$

$$\downarrow \quad (1 + ax + bx^2)^{-1/2}$$

$$\underset{x \ll 1}{\approx} 1 - \frac{a}{2}x + \frac{1}{2}x^2 \left(\frac{3}{4}a^2 - b \right)$$

$$E \approx m_0 c^2 \left[1 - \frac{1}{2} \frac{(Z\alpha)^2}{(n' + j + \frac{1}{2})^2} \right.$$

$$\left. + \frac{1}{2} \frac{(Z\alpha)^4}{(n' + j + \frac{1}{2})^4} \left(\frac{3}{4} - \frac{n' + j + \frac{1}{2}}{j + \frac{1}{2}} \right) \right]$$

INTRODUCE PRINCIPAL QUANTUM NUMBER

$$\underline{\underline{n \equiv n' + j + \frac{1}{2}}}$$

$$n = 1, 2, \dots$$

$$n' = 0, j = \frac{1}{2} \Rightarrow n = 1$$

BINDING ENERGY E_{nj}

$$E_{nj} \equiv E - m_0 c^2 \approx - \frac{1}{2} (m_0 c^2) \frac{(Z\alpha)^2}{n^2} \cdot \left[1 + \frac{(Z\alpha)^2}{n^2} \left(-\frac{3}{4} + \frac{n}{j + \frac{1}{2}} \right) \right]$$

\uparrow
 BOHR THEORY

FINE STRUCTURE

IN DIRAC THEORY BINDING ENERGY
DEPENDS ON n AND j

FOR H ($Z=1$)

$$E_{nj} = - \frac{13.6 \text{ eV}}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(-\frac{3}{4} + \frac{n}{j + \frac{1}{2}} \right) \right]$$

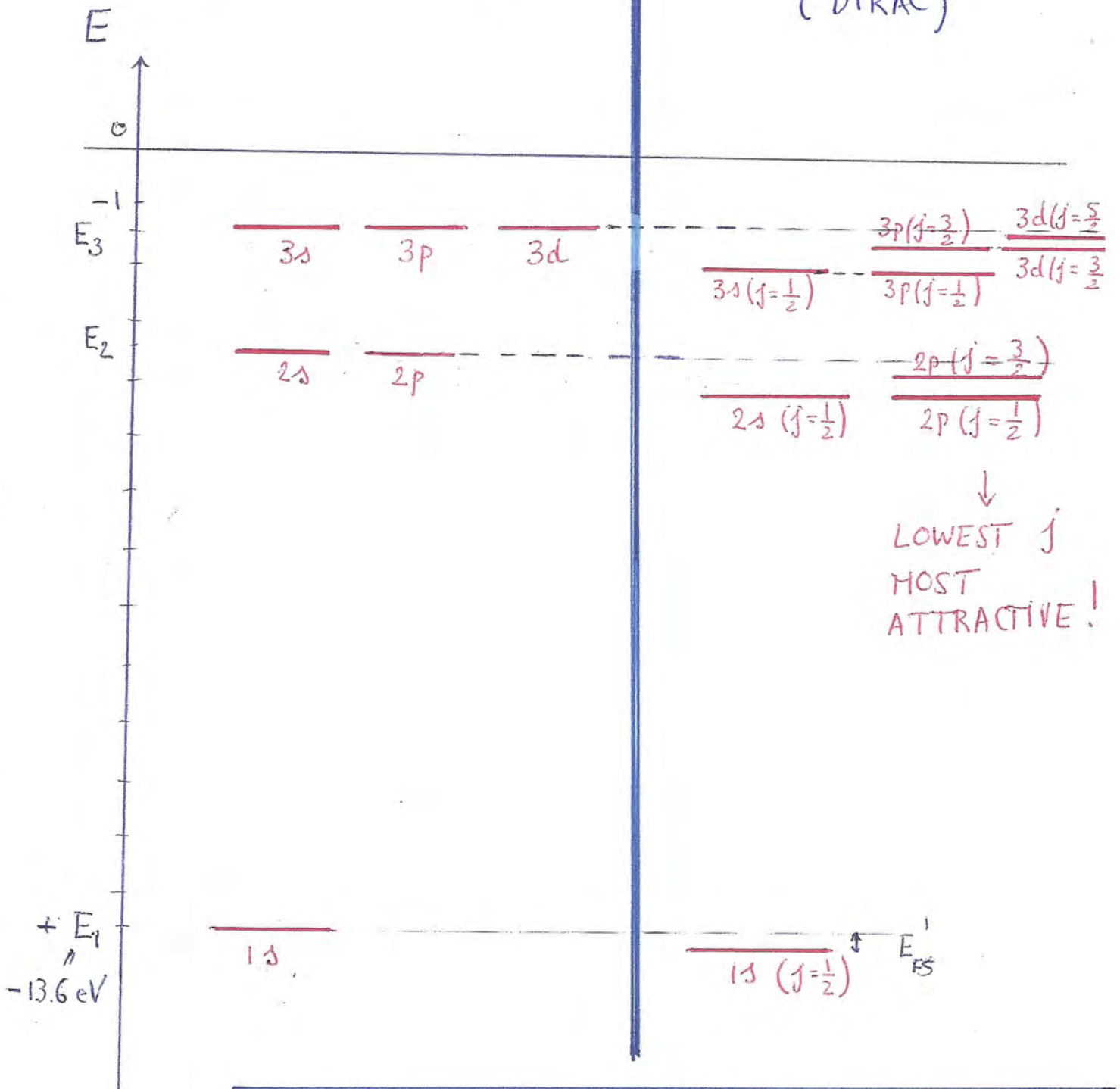
$$= E_n + E_{FS} \quad \text{WITH } E_n = - \frac{13.6 \text{ eV}}{n^2}$$

- $n = 1$
 $j = 1/2$
 $E_{1 \frac{1}{2}} = -13.6 \text{ eV} \left[1 + \frac{\alpha^2}{4} \right] < -13.6 \text{ eV}$
- $n = 2$
 $j = 1/2$
 $E_{2 \frac{1}{2}} = - \frac{13.6 \text{ eV}}{4} \left[1 + \frac{\alpha^2}{4} \cdot \frac{5}{4} \right]$
- $n = 2$
 $j = 3/2$
 $E_{2 \frac{3}{2}} = - \frac{13.6 \text{ eV}}{4} \left[1 + \frac{\alpha^2}{4} \cdot \frac{1}{4} \right]$

FINE - STRUCTURE OF H

H^0 (BOHR MODEL)

$H^0 + H_{FS}^1$
(DIRAC)



↓
LOWEST j
MOST
ATTRACTIVE!

TOTAL
 E

$$E = E_n + E_{FS}^1 = -\frac{13.6 \text{ eV}}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right]$$

DEPENDS ON n & j