QUANTUM THEORY OF RADIATION AND INTERACTION WITH MATTER

1) ABELIAN GAUGE THEORY (QED)

2) QUANTIZATION OF PHOTON FIELD

3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON - RELATIVISTIC)

RAD 1

1) ABELIAN GAUGE THEORY (QED)

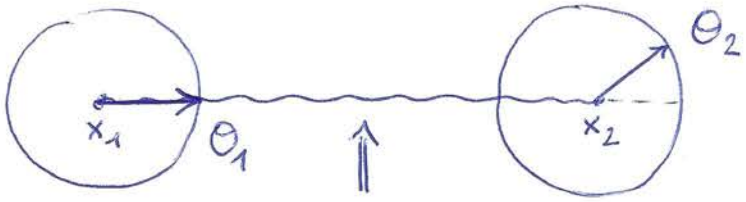
⇒ LOCAL PHASE TRANSFORMATION

↳ FERMION (SPIN 1/2)

$$\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)$$

↳ LOCAL PHASE TF. : U(1) GROUP

$$\psi(x) \xrightarrow{U(1)} e^{i\overset{\text{DEPENDS ON SPACE-TIME POINT}}{\theta(x)}}\psi(x)$$



$$\Delta\theta = \theta_2 - \theta_1$$

SIGNAL WHICH
COMMUNICATES PHASE DIFFERENCE
("PHOTON")

↳ TRANSFORMATION OF $\mathcal{L}_{DIRAC}$ UNDER LOCAL PHASE TF.

$$\partial_\mu \psi(x) \longrightarrow e^{i\theta(x)}\left[\partial_\mu \psi(x) + i(\partial_\mu \theta)\psi\right]$$

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi + i(\partial_\mu\theta)\bar{\psi}\gamma^\mu\psi$$

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\psi$$

$$\mathcal{L}_{DIRAC} \longrightarrow \mathcal{L}_{DIRAC} - (\partial_\mu\theta)\overbrace{\bar{\psi}\gamma^\mu\psi}^{\text{VECTOR}}$$

↳ INVARIANCE UNDER LOCAL PHASE TF.

- $$\mathcal{L}_{DIRAC} \xrightarrow{U(1)} \mathcal{L}_{DIRAC} \underbrace{- (\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi}_{\text{EXTRA TERM}}$$

$\sim (\partial_\mu \theta)$

DIFFERENCE IN PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- TO MAKE $\mathcal{L}$ INVARIANT UNDER LOCAL PHASE TF $\Rightarrow$ INTRODUCE VECTOR FIELD (GAUGE FIELD) WHICH COMPENSATES FOR DIFFERENT CHOICES OF PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- INTRODUCE COVARIANT DERIVATIVE

$$\partial_\mu \underset{\text{REPLACE}}{\Longrightarrow} D_\mu = \partial_\mu + i e A_\mu$$

VECTOR FIELD

ELECTRIC CHARGE
COUPLING OF VECTOR FIELD TO DIRAC FIELD

$$A^\mu \xrightarrow{U(1)} A'^\mu$$

CHOOSE A'^μ SUCH THAT TOTAL $\mathcal{L}$ IS INVARIANT UNDER LOCAL PHASE TF.

- $D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$

$$\xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + i(\partial_\mu \theta) \psi + ie A'_\mu \psi \right]$$

$$= e^{i\theta(x)} \left[\partial_\mu \psi + ie \left(A'_\mu + \frac{1}{e} \partial_\mu \theta \right) \psi \right]$$

CHOOSE $A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \theta$

$$\Downarrow$$

$$D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + ie A_\mu \psi \right]$$

$$D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} D_\mu \psi$$

$$\bar{\psi} i \gamma^\mu D_\mu \psi \xrightarrow{U(1)} \bar{\psi} i \gamma^\mu D_\mu \psi$$

INVARIANT UNDER LOCAL U(1) TF !

$\therefore$ LAGRANGIAN $\mathcal{L} = \bar{\psi} (i \gamma^\mu D_\mu - m) \psi$

IS INVARIANT UNDER LOCAL PHASE TF :

$$\psi(x) \xrightarrow{U(1)} e^{i\theta(x)} \psi(x)$$

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e} \partial^\mu \theta$$

$\Rightarrow$ INTERACTION BETWEEN MATTER & GAUGE FIELD

↳ SYMMERY OF $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ UNDER LOCAL $U(1)$ IS CALLED ABELIAN GAUGE SYMMETRY

$\downarrow$

REFERS TO GAUGE GROUP $U(1)$

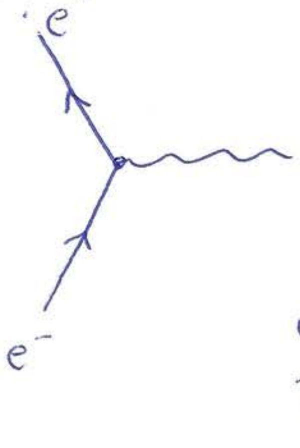
↳ COUPLING TO GAUGE (PHOTON) FIELD

$$
\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\
&= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e(\bar{\psi}\gamma^\mu\psi)A_\mu \\
&= \mathcal{L}_{DIRAC} + \mathcal{L}_{INT}
\end{aligned}
$$

$\uparrow$ INTERACTION LAGRANGIAN

$$
\boxed{\mathcal{L}_{INT} = -e(\bar{\psi}\gamma^\mu\psi)A_\mu}
$$

GRAPHICALLY:



γ (DESCRIBED BY A_μ FIELD).

e IS STRENGTH OF COUPLING BETWEEN FIELDS

⇒ QUANTUM ELECTRODYNAMICS (QED)

↳ INTERACTION BETWEEN MATTER (SPIN 1/2) AND GAUGE FIELDS (PHOTONS) IS DESCRIBED BY

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

- FREE MATTER FIELD ψ

$$\mathcal{L}_{DIRAC} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- INTERACTION BETWEEN MATTER & GAUGE FIELDS

$$\mathcal{L}_{INT} = -e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\equiv -\underbrace{J^\mu_{em}(x)}_{\text{ELECTROMAGNETIC CURRENT}} A_\mu$$

$$J^\mu_{em} = e\bar{\psi}\gamma^\mu\psi$$

- FREE PHOTON FIELD

FIELD TENSOR $\underline{\underline{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu}}$.

UNDER $U(1)$ $\quad A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu\theta$

$$F^{\mu\nu} \xrightarrow{U(1)} F^{\mu\nu}$$

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

↳ $\mathcal{L}_{QED}$ IS INVARIANT UNDER LOCAL $U(1)$!

$$\psi(x) \xrightarrow{U(1)} e^{i\theta(x)} \psi(x)$$

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \theta$$

$U(1)$ GAUGE SYMMETRY

↳ FIELD EQUATIONS FOR A^μ

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0$$

$$\downarrow \quad \text{FOR } \phi_r = A_\nu$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = - J^\nu_{em}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - \frac{1}{2} \cdot 2 \cdot F^{\mu\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu_{em}}$$

INHOMOGENEOUS MAXWELL EQ !

↳ EM CURRENT : SOURCE TERM IN MAXWELL EQ.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu_{em}$$

↳ GAUGE INVARIANCE

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \theta$$

FREEDOM TO CONSTRAIN A^μ

POSSIBLE CHOICES OF GAUGE ARE

- $\partial_\mu A^\mu = 0$: LORENZ GAUGE (COVARIANT)
- $\overline{\nabla} . \overline{A} = 0$ $(A^0 = 0)$: COULOMB GAUGE (NON - COVARIANT)
  ↳ USED FOR FREE FIELDS
  ($J^\nu_{em} = 0$, ABSENCE OF SOURCES)
- $n_\mu A^\mu = 0$: AXIAL GAUGE
  WITH $n_\mu n^\mu = -1$
  e.g. $n^\mu (0, 0, 0, 1)$

IN FOLLOWING WE WILL OFTEN USE
LORENZ GAUGE $\partial_\mu A^\mu = 0$

↓

$$\Box A^\nu = J^\nu_{em}$$

$$\partial_\mu A^\mu = 0 \qquad \text{CONSTRAINT}$$

GAUGE INVARIANCE ENSURES THAT RESULTS FOR PHYSICAL OBSERVABLES ARE INVARIANT UNDER CHOICE OF GAUGE

2) QUANTIZATION OF PHOTON FIELD

⟹ PRESENCE OF CONSTRAINTS : ISSUES

↳ $$\Box A^\nu = J^\nu_{em}$$

$$\partial_\nu A^\nu = 0 \qquad \text{(LORENZ GAUGE)}$$

HOW TO QUANTISE A^ν IN PRESENCE OF CONSTRAINT $\partial_\nu A^\nu = 0$

↳ ISSUE 1 : CANONICAL MOMENTA

$$A^\nu \longrightarrow \Pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu}$$

$$\Pi^0 = 0$$

$$\Pi^i = -F^{0i} = -\partial^0 A^i + \partial^i A^0$$

$$= \left(-\bar{\nabla} A^0 - \frac{\partial \bar{A}}{\partial t}\right)^i$$

$$= \bar{E}^i \quad \text{(ELECTRIC FIELD)}$$

FOR A^1, A^2, A^3 OK

FOR A^0 ⟹ WE CANNOT IMPOSE COMMUTATION RELATIONS BECAUSE $\Pi^0 = 0$

⇒ GAUGE FIXING

↳ TO SOLVE ABOVE PROBLEMS WHEN QUANTIZING THE ABELIAN GAUGE THEORY CONSIDER ALTERNATIVE $\mathcal{L}$

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ DESCRIBES FREE PHOTON FIELD

$$\mathcal{L}_{em} \longrightarrow \boxed{\mathcal{L}'_{em} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\mathcal{L}_{em}} \underbrace{- \frac{1}{2} (\partial_\mu A^\mu)^2}_{\mathcal{L}_{GF}}}$$

(GAUGE FIXING)

↳ EQUATIONS OF MOTION

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - F^{\mu\nu} - (\partial_\alpha A^\alpha)\, g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu (\partial_\alpha A^\alpha) = 0$$

$$\Box A^\nu - \partial^\nu (\cancel{\partial_\mu A^\mu}) + \partial^\nu(\cancel{\partial_\alpha A^\alpha}) = 0$$

$$\Box A^\nu = 0$$ ↷ FIELD EQ. CORRESPONDING WITH $\mathcal{L}'_{em}$ CORRESPOND WITH CHOICE OF LORENZ GAUGE

↳ CANONICAL MOMENTA

$$A^\nu \to \Pi^\nu = - F^{0\nu} - (\partial_\alpha A^\alpha)\, g^{0\nu}$$

$$\begin{cases} \Pi^0 = - \partial_\alpha A^\alpha \\ \Pi^i = - F^{0i} \end{cases}$$

, OK ISSUE 1 IF $\partial_\alpha A^\alpha$ CONSIDERED AS OPERATOR

↳ EQUAL TIME COMMUTATION RELATIONS (ETCR)

TO QUANTIZE THE THEORY, WE IMPOSE

$$[A^\mu(\bar{x},t), A^\nu(\bar{x}',t)]_- = 0$$

$$[\Pi^\mu(\bar{x},t), \Pi^\nu(\bar{x}',t)]_- = 0$$

$$[A^\mu(\bar{x},t), \Pi^\nu(\bar{x}',t)]_- = i g^{\mu\nu} \delta^3(\bar{x}-\bar{x}')$$

WITH $[A,B]_- = AB - BA$ COMMUTATOR

SPIN 1 FIELD DESCRIBES BOSONS (PHOTONS)

A^μ, Π^μ CONSIDERED AS OPERATORS

$$\rightsquigarrow [A^\mu(\bar{x},t), \Pi^0(\bar{x}',t)]_- = i g^{\mu 0} \delta^3(\bar{x}-\bar{x}')$$

$$= -[A^\mu(\bar{x},t), \dot{A}^0(\bar{x}',t)]_- - \bar{\nabla}_{x'} \cdot \underbrace{\cancel{[A^\mu(\bar{x},t), \bar{A}(\bar{x}',t)]_-}}_{0}$$

$$\rightsquigarrow \quad [A^\mu(\bar{x},t),\ \underbrace{\Pi^i(\bar{x}',t)}_{-\dot{A}^i + \partial^i A^0}]_- = i g^{\mu i}\,\delta^3(\bar{x}-\bar{x}')$$

$$= -[A^\mu(\bar{x},t),\ \dot{A}^i(\bar{x}',t)]_- - \bar{\nabla}^i_{x'} \cdot \underbrace{\cancel{[A^\mu(\bar{x},t),\ A^0(\bar{x}',t)]_-}}_{0}$$

$$\therefore \quad \| \; [A^\mu(\bar{x},t),\ \dot{A}^\nu(\bar{x}',t)]_- = -i g^{\mu\nu}\,\delta^3(\bar{x}-\bar{x}')$$

$\hookrightarrow \mu = 0: \quad \boxed{[A^\mu(\bar{x},t),\ \dot{A}^0(\bar{x}',t)]_- = -i\,\delta^3(\bar{x}-\bar{x}')}$

$\hookrightarrow \mu = i: \quad \boxed{[A^\mu(\bar{x},t),\ \dot{A}^i(\bar{x}',t)]_- = +i\,\delta^3(\bar{x}-\bar{x}')}$

$\hookrightarrow$ PHOTON PROPAGATOR (IN LORENZ GAUGE)

$$\Box A^\mu = 0$$

$$\underbrace{(-k^2 g^{\mu\nu})}_{\text{INVERSE EXISTS}} A_\nu = 0$$

$$(-k^2 g^{\mu\nu}) \cdot (A\, g_{\nu\lambda} + B\, k_\nu k_\lambda) = g^\mu{}_\lambda$$

$$A = -\frac{1}{k^2} \qquad \Rightarrow \qquad \text{PHOTON PROPAGATOR} \sim \frac{-g^{\mu\nu}}{k^2 + i\varepsilon} \quad \nu \overset{k}{\rightsquigarrow} \mu$$

$$B = 0$$

⟹ NORMAL MODE EXPANSION OF $A^\mu(\bar{x},t)$

↳ UPON QUANTIZATION

CLASSICAL FIELDS A^μ

ARE RE-INTERPRETED AS FIELD OPERATORS $\hat{A}^\mu$

THAT SATISFY ETCR

(NOTE: FOR SIMPLICITY WE WILL DROP ^ NOTATION IN QFT THE FIELDS ARE UNDERSTOOD AS OPERATORS)

↳ POLARIZATION VECTORS:

A NORMAL MODE SOLUTION IS CHARACTERIZED

BY $\sim e^{ik.x}\, \varepsilon^\mu(\bar{k},\lambda) \qquad \lambda = 0,1,2,3$

WITH $k(\omega_{\bar{k}}, \underbrace{0,0,|\bar{k}|}_{\bar{k}})$ CONVENIENT CHOICE $\bar{k} = |\bar{k}|\,\bar{e}_z$

$\omega_{\bar{k}} = |\bar{k}|$

$$\varepsilon^\mu(\bar{k},\lambda=0) = (1,0,0,0) \qquad \text{SCALAR POL.}$$

$$\left.\begin{aligned}\varepsilon^\mu(\bar{k},\lambda=1) &= (0,1,0,0)\\ \varepsilon^\mu(\bar{k},\lambda=2) &= (0,0,1,0)\end{aligned}\right\} \text{TRANSVERSE POL.} \quad 0 = \bar{k}\cdot\bar{\varepsilon}(\bar{k},\lambda=\tfrac{1}{2})$$

$$\varepsilon^\mu(\bar{k},\lambda=3) = (0,0,0,1) \qquad \text{LONGITUDINAL POL.}$$

NORMALIZATION
$$\varepsilon^\mu(\bar{k},\lambda)\,\varepsilon^*_\mu(\bar{k},\lambda') = -\zeta_\lambda\,\delta_{\lambda\lambda'}$$
$$(\zeta_0 \equiv -1,\ \zeta_i \equiv +1)$$

↳ NORMAL MODE EXPANSION

$$\boxed{A^\mu(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} \left\{ a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\,e^{-ik.x} + a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\,e^{+ik.x} \right\}}$$

$\omega_{\bar{k}} = |\bar{k}|$ PARTICLE WITH MASS 0 (PHOTON)
(→ MOVES WITH SPEED OF LIGHT)

$a(\bar{k},\lambda)$ ANNIHILATES PHOTON WITH MOMENTUM $\bar{k}$
& POLARIZATION λ

$a^+(\bar{k},\lambda)$ CREATES PHOTON " "

⟹ COMMUTATION RELATIONS FOR a, a^+

↳ $$\int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) + \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\; e^{-i\omega_{\bar{k}}t}$$

↳ $$\int d^3\bar{x}\; e^{+i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) - \frac{i}{\omega_{\bar{k}}} \dot{A}^\mu(\bar{x},t) \right\}$$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\; e^{i\omega_{\bar{k}}t}$$

$\downarrow$ USING $\varepsilon^{\mu}(\bar{k},\lambda)\,\varepsilon^{*}_{\mu}(\bar{k},\lambda') = -\zeta_{\lambda}\,\delta_{\lambda\lambda'}$

$$a(\bar{k},\lambda) = -\zeta_{\lambda}\,\frac{1}{2}\sqrt{2\omega_{\bar{k}}}\;e^{+i\omega_{\bar{k}}t}\int d^3\bar{x}\;e^{-i\bar{k}.\bar{x}}\,\varepsilon^{*}_{\mu}(\bar{k},\lambda)\left\{A^{\mu}+\frac{i}{\omega_{\bar{k}}}\mathring{A}^{\mu}\right\}$$

$$a^{+}(\bar{k},\lambda) = -\zeta_{\lambda}\,\frac{1}{2}\sqrt{2\omega_{\bar{k}}}\;e^{-i\omega_{\bar{k}}t}\int d^3\bar{x}\;e^{+i\bar{k}.\bar{x}}\,\varepsilon_{\mu}(\bar{k},\lambda)\left\{A^{\mu}-\frac{i}{\omega_{\bar{k}}}\mathring{A}^{\mu}\right\}$$

$\hookrightarrow\;\left[a(\bar{k},\lambda),\,a^{+}(\bar{k}',\lambda')\right]_{-}$

$$= \zeta_{\lambda}\zeta_{\lambda'}\,\frac{1}{4}(2\omega_{\bar{k}})^{1/2}(2\omega_{\bar{k}'})^{1/2}e^{i(\omega_{\bar{k}}-\omega_{\bar{k}'})t}\,\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon_{\nu}(\bar{k}',\lambda')$$

$$\cdot\int d^3\bar{x}\int d^3\bar{x}'\;e^{-i\bar{k}.\bar{x}}\,e^{+i\bar{k}'.\bar{x}'}$$

$$\cdot\underbrace{\left[A^{\mu}(\bar{x},t)+\frac{i}{\omega_{\bar{k}}}\mathring{A}^{\mu}(\bar{x},t),\;A^{\nu}(\bar{x}',t)-\frac{i}{\omega_{\bar{k}'}}\mathring{A}^{\nu}(\bar{x}',t)\right]_{-}}_{-\frac{1}{\omega_{\bar{k}'}}g^{\mu\nu}\delta^3(\bar{x}-\bar{x}')\;-\;\frac{1}{\omega_{\bar{k}}}g^{\mu\nu}\delta^3(\bar{x}-\bar{x}')}$$

$$= \zeta_{\lambda}\zeta_{\lambda'}\,\frac{1}{4}(2\omega_{\bar{k}})^{1/2}(2\omega_{\bar{k}'})^{1/2}(-1)\left(\frac{1}{\omega_{\bar{k}}}+\frac{1}{\omega_{\bar{k}'}}\right)e^{i(\omega_{\bar{k}}-\omega_{\bar{k}'})t}$$

$$\cdot\,\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon^{\mu}(\bar{k}',\lambda')\underbrace{\int d^3\bar{x}\;e^{-i(\bar{k}-\bar{k}').\bar{x}}}_{(2\pi)^3\delta^3(\bar{k}-\bar{k}')}$$

$$[\, a(\bar{k},\lambda)\,,\; a^{+}(\bar{k}',\lambda')\,]_{-}$$

$$= -\,\zeta_\lambda\,\zeta_{\lambda'}\;\frac{1}{4}\,(2\omega_{\bar{k}})\left(\frac{2}{\omega_{\bar{k}}}\right)\cdot(2\pi)^3\;\underbrace{\varepsilon^{*}_{\mu}(\bar{k},\lambda)\,\varepsilon^{\mu}(\bar{k},\lambda')}_{-\,\zeta_\lambda\,\delta_{\lambda\lambda'}}$$

$$\downarrow \quad \zeta_\lambda^2 = 1$$

$$= \zeta_\lambda\;\delta_{\lambda\lambda'}\;(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

$\therefore$

$$\boxed{\begin{array}{l} [\, a(\bar{k},\lambda)\,,\; a^{+}(\bar{k}',\lambda')\,]_{-} = \zeta_\lambda\,\delta_{\lambda\lambda'}\,(2\pi)^3\,\delta^3(\bar{k}-\bar{k}') \\ \text{ANALOGOUSLY} \\ [\, a(\bar{k},\lambda)\,,\; a(\bar{k}',\lambda')\,]_{-} = 0 \\ [\, a^{+}(\bar{k},\lambda)\,,\; a^{+}(\bar{k}',\lambda')\,]_{-} = 0 \end{array}}$$

$$\zeta_0 = -1\,,\quad \zeta_i = +1 \quad (i = 1,2,3)$$

FOR $\lambda = 1, 2, 3$

$$[\, a(\bar{k},\lambda)\,,\; a^{+}(\bar{k}',\lambda)\,]_{-} = \delta_{\lambda\lambda'}\;(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 0$

$$[\, a(\bar{k},0)\,,\; a^{+}(\bar{k}',0)\,]_{-} = -\;(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 1,2,3$ $\longrightarrow$ STANDARD BOSON COMMUTATION RELATIONS

$\lambda = 0$ $\longrightarrow$ SIGN CHANGE

↳ IF WE CONSIDER VACUUM STATE AS STATE WITHOUT PHOTONS

i.e. $a(\bar{k},\lambda)\,|0\rangle = 0 \qquad \lambda = 0,1,2,3$

$\Downarrow$

1 - PHOTON STATE WITH POLARIZATION λ

$$|1\gamma,\lambda\rangle = \int d^3\bar{k}\; f(\bar{k})\; a^{+}(\bar{k},\lambda)\,|0\rangle$$

↳ WAVE PACKET

$$\int d^3\bar{k}\; |f(\bar{k})|^2 < \infty$$

NORMALIZATION OF 1γ STATE

$$\langle 1\gamma,\lambda\,|\,1\gamma,\lambda\rangle$$

$$= \int d^3\bar{k} \int d^3\bar{k}'\; f^{*}(\bar{k})\, f(\bar{k}')\; \underbrace{\langle 0|\, a(\bar{k},\lambda)\, a^{+}(\bar{k}',\lambda)\,|0\rangle}_{\zeta_\lambda\,(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')}$$

$$= \zeta_\lambda\,(2\pi)^3 \int d^3\bar{k}\; |f(\bar{k})|^2$$

$$= \begin{cases} > 0 & \text{FOR } \lambda = 1,2,3 \\ < 0 & \text{FOR } \lambda = 0 \end{cases}$$

NEGATIVE NORM STATE

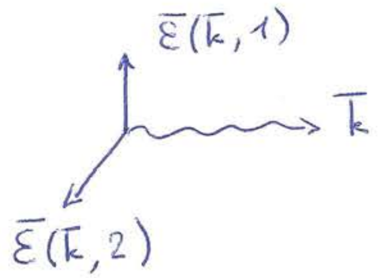
↳ HAMILTONIAN :

$$H = \int \frac{d^3\bar{k}}{(2\pi)^3}\, \omega_{\bar{k}} \left\{ - a^+(\bar{k},0)\, a(\bar{k},0) + \sum_{i=1}^{3} a^+(\bar{k},i)\, a(\bar{k},i) \right\}$$

STATES WITH $\lambda = 0$ LEAD TO NEGATIVE ENERGY !

↳ PHYSICAL STATES :

FREE MAXWELL FIELD HAS ONLY 2 TRANSVERSE COMPONENTS



↳ $\lambda = 0$, $\lambda = 3$ STATES SHOULD NOT APPEAR IN PHYSICAL STATES UPON QUANTIZATION

$|\Psi\rangle$ PHYSICAL VACUUM (HAS NO TRANSVERSE PHOTONS)

WE REQUIRE

$$\underline{\underline{\partial_\mu A^{\mu(+)} |\Psi\rangle = 0}} \qquad \text{(GUPTA - BLEULER)}$$

(+) STANDS FOR POS. FREQUENCY (ANNIHILATING) PART IN A^μ

NOTE : THIS IS A WEAKER CONDITION THAN CONSIDERING $\partial_\mu \hat{A}^\mu = 0$ AS OPERATOR CONDITION

$$\partial_\mu A^{\mu(+)} \, | \Psi \rangle = 0$$

$$\downarrow$$

$$\partial_\mu \int \frac{d^3 \bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda a(\bar{k},\lambda) \, \varepsilon^\mu(\bar{k},\lambda) \, e^{-ik.x} \, | \Psi \rangle = 0$$

$$\int \frac{d^3 \bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda (-i) \, a(\bar{k},\lambda) \, e^{-ik.x} \, k_\mu \, \varepsilon^\mu(\bar{k},\lambda) \, | \Psi \rangle = 0$$

$$\downarrow$$

$$\begin{aligned} & k_\mu \, \varepsilon^\mu(\bar{k},\lambda) \\ &= \omega_{\bar{k}} \, \varepsilon^0(\bar{k},\lambda) - |\bar{k}| \, \varepsilon^3(\bar{k},\lambda) \\ &= \omega_{\bar{k}} \, (\delta_{\lambda 0} - \delta_{\lambda 3}) \end{aligned}$$

$$\forall \bar{k} : \quad \underline{\underline{\big(a(\bar{k},0) - a(\bar{k},3) \big) \, | \Psi \rangle = 0.}}$$

$$\Downarrow$$

$$\langle \Psi | a^\dagger(\bar{k},3) \, a(\bar{k},3) | \Psi \rangle = \langle \Psi | a^\dagger(\bar{k},0) \, a(\bar{k},0) | \Psi \rangle$$

- PHYSICAL VACUUM STATE HAS EQUAL NUMBER OF SCALAR ($\lambda = 0$) AND LONGITUDINAL ($\lambda = 3$) PHOTONS

- THE COMBINED ENERGY OF SCALAR & LONGITUDINAL PHOTONS IS ZERO

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3\bar{k}}{(2\pi)^3} \, \omega_{\bar{k}} \sum_{\lambda=1,2} \langle \Psi | a^{+}(\bar{k},\lambda) a(\bar{k},\lambda) | \Psi \rangle$$

- PHYSICAL QUANTITIES ONLY INVOLVE TRANSVERSE PHOTONS ($\lambda = 1, 2$).

- ALTERING THE ALLOWED ADMIXTURES OF SCALAR AND LONGITUDINAL PHOTONS IS EQUIVALENT TO A GAUGE TF. BETWEEN 2 POTENTIALS, BOTH OF WHICH ARE IN LORENZ GAUGE

  ($\rightarrow$ EXERCISE)

3) APPLICATIONS OF INTERACTION OF E.M. FIELD WITH MATTER (NON-RELATIVISTIC)

- HAMILTONIANS

↳ MANY-BODY SYSTEM (MATTER) $H_M = \sum_{i=1}^{N} \frac{\bar{P}_i^2}{2m_i} + V$

↳ RADIATION FIELD $H_{EM} = \frac{1}{2} \int d^3\bar{x} \left(\bar{E} \cdot \bar{E}^* + \bar{B} \cdot \bar{B}^* \right)$

↳ INTERACTION HAS TO RESPECT GAUGE-INVARIANCE

$$\psi(\bar{x},t) \longrightarrow e^{\frac{ie}{\hbar c} \chi(\bar{x},t)} \psi(\bar{x},t)$$

LOCAL PHASE TRANSFORMATION HAS TO LEAVE THEORY INVARIANT

$$\hat{p}\, \psi(\bar{x},t) = -i\hbar \bar{\nabla} \psi(\bar{x},t)$$

$$\longrightarrow e^{\frac{ie}{\hbar c} \chi(\bar{x},t)} \left[-i\hbar \bar{\nabla} \psi + \frac{e}{c} \left(\bar{\nabla} \chi \right) \psi \right]$$

KINETIC ENERGY TERM CAN BE MADE INVARIANT BY MINIMAL SUBSTITUTION

$$\boxed{\bar{P}_i \Rightarrow \bar{P}_i - \frac{e_i}{c} \bar{A}(x_i)}$$

UNDER GAUGE TRANSFORMATION, $\bar{A}$ TRANSFORMS AS

$$\bar{A} \rightarrow \bar{A} + \bar{\nabla}\chi$$

$$\therefore \quad \left(\bar{p} - \frac{e}{c}\bar{A}\right)\psi(\bar{x},t)$$

$$\rightarrow e^{\frac{ie}{\hbar c}\chi(\bar{x},t)} \left\{ -i\hbar\bar{\nabla} + \cancel{\frac{e}{c}(\bar{\nabla}\chi)} - \frac{e}{c}\bar{A} - \cancel{\frac{e}{c}(\bar{\nabla}\chi)} \right\}\psi$$

$$= e^{\frac{ie}{\hbar c}\chi(\bar{x},t)} \left(\underbrace{-i\hbar\bar{\nabla}}_{\bar{p}} - \frac{e}{c}\bar{A}\right)\psi$$

$\therefore$ HAMILTONIAN IS GAUGE INVARIANT

$\Downarrow$

EXPECTATION VALUES REMAIN UNCHANGED

$$\text{e.g.} \quad \int d^3\bar{x} \; \psi^*(\bar{x},t)\frac{1}{2m}\left(-i\hbar\bar{\nabla} - \frac{e}{c}\bar{A}\right)^2 \psi(\bar{x},t)$$

$$\boxed{\begin{aligned} H_{tot} &= \sum_{i=1}^{N} \frac{1}{2m_i}\left(\bar{p}_i - \frac{e_i}{c}\bar{A}_i\right)^2 + V + H_{EM} \\ &= H_M + H_{INT} + H_{EM} \end{aligned}}$$

$\uparrow$ MATTER

$\uparrow$ INTERACTION RADIATION WITH MATTER

↳ INTERACTION HAMILTONIAN

$$\boxed{\begin{aligned} H_{INT} &= \sum_i \left\{ -\frac{e_i}{m_i c}\, \bar{P}_i \cdot \bar{A}(\bar{x}_i) + \frac{e_i^2}{2 m_i c^2}\, \bar{A}^2(\bar{x}_i) \right\} \\ &\equiv H'_{INT} + H''_{INT} \end{aligned}}$$

LINEAR IN $\bar{A}$ QUADRATIC IN $\bar{A}$

(NOTE IN H'_{INT} : $\bar{P}_i \cdot \bar{A} + \bar{A} \cdot \bar{P}_i = 2\, \bar{P}_i \cdot \bar{A}$ (IN COULOMB GAUGE $\bar{\nabla} \cdot \bar{A} = 0$)

- INTERACTION HAMILTONIAN IN SECOND QUANTIZATION

↳ RECALL : $\bar{A}(\bar{x}) = \sum_{\bar{k}} \sum_{\sigma=1,2} N_k\, \varepsilon_{\bar{k}\sigma} \left\{ \hat{a}_{\bar{k}\sigma}\, e^{i \bar{k} \cdot \bar{x}} + \hat{a}^+_{\bar{k}\sigma}\, e^{-i \bar{k} \cdot \bar{x}} \right\}$

WITH $N_k = \left(\dfrac{\hbar c^2}{2 \omega_k L^3} \right)^{1/2}$

↳ INTERACTION OF 1 PARTICLE WITH RADIATION FIELD

$$\hat{H}'_{INT} = -\frac{e}{mc} \sum_{\bar{k}} \sum_{\sigma} N_{\bar{k}}\, \hat{\bar{P}} \cdot \varepsilon_{\bar{k}\sigma} \left\{ \hat{a}_{\bar{k}\sigma}\, e^{i \bar{k} \cdot \bar{x}} + \hat{a}^+_{\bar{k}\sigma}\, e^{-i \bar{k} \cdot \bar{x}} \right\}$$

$$\hat{H}''_{INT} = \frac{e^2}{2mc^2} \sum_{\bar{k}\sigma} \sum_{\bar{k}'\sigma'} \left(\frac{\hbar c^2}{2L^3} \right) \frac{\bar{\varepsilon}_{\bar{k}\sigma} \cdot \bar{\varepsilon}_{\bar{k}'\sigma'}}{(\omega_k \, \omega_{k'})^{1/2}}$$

$$\cdot \Big\{ \hat{a}_{\bar{k}\sigma} \, \hat{a}_{\bar{k}'\sigma'} \, e^{i(\bar{k}+\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}_{\bar{k}\sigma} \, \hat{a}^{+}_{\bar{k}'\sigma'} \, e^{i(\bar{k}-\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}^{+}_{\bar{k}\sigma} \, \hat{a}_{\bar{k}'\sigma'} \, e^{i(-\bar{k}+\bar{k}')\cdot\bar{x}}$$

$$+ \hat{a}^{+}_{\bar{k}\sigma} \, \hat{a}^{+}_{\bar{k}'\sigma'} \, e^{-i(\bar{k}+\bar{k}')\cdot\bar{x}} \Big\}$$

↳ WE WILL CONSIDER $H'_{INT} + H''_{INT}$ AS PERTURBATION TO $H_M + H_{EM}$

UNPERTURBED STATES

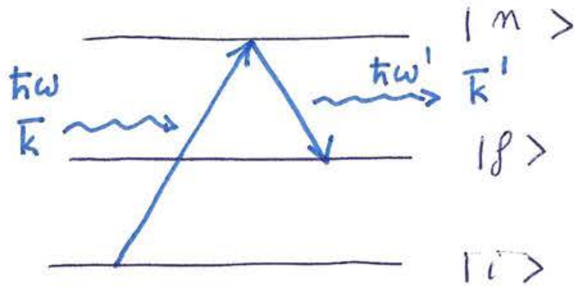
$$|\Psi_0\rangle = \underbrace{|\ldots n_i \ldots\rangle_M}_{\text{MATTER}} \otimes \underbrace{|\ldots n_{\bar{k}\sigma} \ldots\rangle_{EM}}_{\text{RADIATION FIELD}}$$

$H'_{INT} + H''_{INT}$ WILL INDUCE TRANSITIONS BETWEEN THESE UNPERTURBED STATES.

↳ EXAMPLES : PHOTONS ARE ABSORBED ($\hat{a}_{\bar{k}\sigma}$)

AND/OR EMITTED ($\hat{a}^{+}_{\bar{k}\sigma}$)

1)



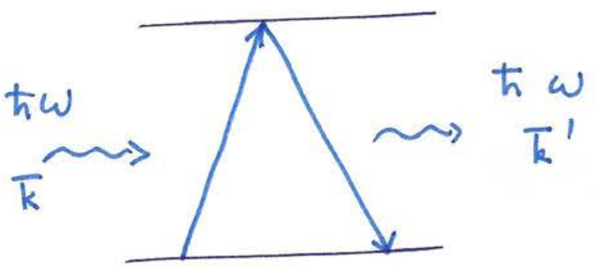
INELASTIC

PHOTON SCATTERING

(RAMAN SCATTERING)

$\omega' \neq \omega$, $\bar{k}' \neq \bar{k}$

2) IF $|i\rangle = |f\rangle$



ELASTIC

PHOTON SCATTERING

$\omega' = \omega$, $|\bar{k}'| = |\bar{k}|$

BUT DIRECTIONS

CAN BE DIFFERENT

• INTERLUDE : TIME - DEPENDENT PERTURBATION THEORY

↳ QUANTUM SYSTEM DESCRIBED BY $\hat{H}_0$ (UNPERTURBED HAMILT.)

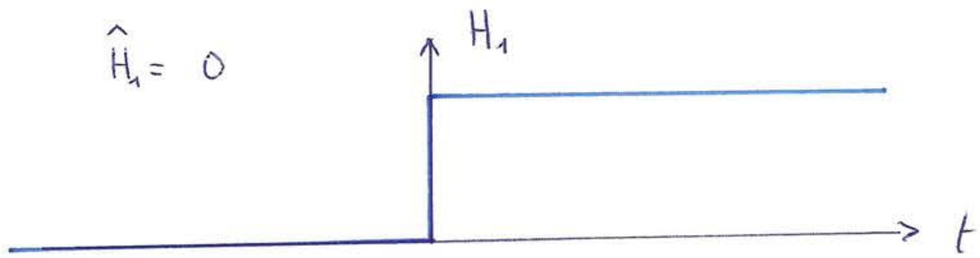
$$\hat{H}_0 \, | \Psi_0(t) \rangle = i\hbar \frac{\partial}{\partial t} | \Psi_0(t) \rangle$$

$$| \Psi_0(t) \rangle = \sum_n a_n \, e^{-\frac{i}{\hbar} E_n t} \, | \psi_n \rangle$$

$$\hat{H}_0 \, | \psi_n \rangle = E_n \, | \psi_n \rangle$$

↑ EIGENSTATES ↑ EIGENVALUES.

↳ AT $t = 0$, PERTURBATION $\hat{H}_1$ IS SWITCHED ON



$t > 0$: $\hat{H} = \hat{H}_0 + \hat{H}_1$

$$\left(\hat{H}_0 + \hat{H}_1 \right) | \Psi(t) \rangle = i\hbar \frac{\partial}{\partial t} | \Psi(t) \rangle$$

EXPAND $| \Psi(t) \rangle$ IN EIGENSTATES OF $\hat{H}_0$

$$| \Psi(t) \rangle = \sum_n a_n(t) \, e^{-\frac{i}{\hbar} E_n t} \, | \psi_n \rangle$$

⇑ $a_n(t)$ VARIES NOW WITH TIME DUE TO $\hat{H}_1$

$$i\hbar \frac{\partial}{\partial t} \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t} |\psi_n\rangle$$

$$= (\hat{H}_0 + \hat{H}_1) \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t} |\psi_n\rangle$$

$$\Downarrow$$

$$\sum_n \cancel{a_n(t)\, E_n\, e^{-\frac{i}{\hbar}E_n t}} |\psi_n\rangle + \sum_n \left(i\hbar \frac{\partial a_n}{\partial t}\right) e^{-\frac{i}{\hbar}E_n t} |\psi_n\rangle$$

$$= \sum_n \cancel{a_n(t)\, E_n\, e^{-\frac{i}{\hbar}E_n t}} |\psi_n\rangle + \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t}\, \hat{H}_1 |\psi_n\rangle$$

$$\Downarrow \qquad \langle\psi_m|$$

$$i\hbar \left(\frac{d}{dt} a_m(t)\right) e^{-\frac{i}{\hbar}E_m t} = \sum_n a_n(t)\, e^{-\frac{i}{\hbar}E_n t} \langle\psi_m|\hat{H}_1|\psi_n\rangle$$

$$\boxed{i\hbar \frac{da_m}{dt} = \sum_n a_n(t) \langle\psi_m|\hat{H}_1|\psi_n\rangle\, e^{+\frac{i}{\hbar}(E_m - E_n)t}}$$

↳ ASSUME AT $t \leq 0$ SYSTEM IS IN STATE $|\psi_n\rangle$

$a_n(t \leq 0) = 1$, $a_m(t \leq 0) = 0$ FOR $m \neq n$

FOR $\hat{H}_1 \ll \hat{H}_0$ APPLY 1° ORDER PERTURBATION THEORY

i.e. ON RHS APPROXIMATE $a_n(t > 0) \approx a_n(t = 0) = 1$.

$a_m(t > 0) \approx a_m(t = 0) = 0,\ m \neq n$

$$\therefore \quad i\hbar \frac{d}{dt} a_m \approx \langle \psi_m | \hat{H}_1 | \psi_n \rangle \, e^{\frac{i}{\hbar}(E_m - E_n)t}$$

$$\boxed{a_m(t) \approx -\frac{i}{\hbar} \int_{t'=0}^{t'=t} dt' \, \langle \psi_m | \hat{H}_1 | \psi_n \rangle \, e^{\frac{i}{\hbar}(E_m - E_n)t'}}$$

$$\Downarrow$$

↳ IF $\hat{H}_1$ DOES NOT DEPEND ON TIME

$$a_m(t) \approx -\frac{i}{\hbar} \langle \psi_m | \hat{H}_1 | \psi_n \rangle \int_{t'=0}^{t'=t} dt' \, e^{\frac{i}{\hbar}(E_m - E_n)t'}$$

$$\boxed{a_m(t) \approx + \frac{1}{(E_n - E_m)} \langle \psi_m | \hat{H}_1 | \psi_n \rangle \left(e^{\frac{i}{\hbar}(E_m - E_n)t} - 1 \right)}$$

TRANSITION PROBABILITY FROM $n \to m$ IN 1° ORDER P.T.

$$\boxed{P_{n \to m}(t) = |a_m(t)|^2 = \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2 \frac{\sin^2\left(\frac{E_m - E_n}{2\hbar} t \right)}{\left(\frac{E_n - E_m}{2} \right)^2}}$$

FERMI'S GOLDEN RULE

⇝ TAKE LONG TIME LIMIT $t \to \infty$

USE
$$\left. \frac{\sin^2(xt)}{\pi t x^2} \right|_{t\to\infty} = \delta(x)$$

$$x \neq 0 \Rightarrow \left. \frac{\sin^2(xt)}{\pi t x^2} \right|_{t\to\infty} \longrightarrow 0$$

$$\int_{-\infty}^{+\infty} dx \, \frac{\sin^2(xt)}{\pi t x^2} = \frac{2}{\pi} \int_0^\infty d(tx) \, \frac{\sin^2(xt)}{(xt)^2} = 1$$

$$\rightsquigarrow P_{n\to m}(t \gg) \simeq |\langle \psi_m | \hat{H}_1 | \psi_n \rangle|^2 \frac{\pi t}{\hbar^2} \underbrace{\delta\left(\frac{E_m - E_n}{2\hbar}\right)}_{2\hbar\,\delta(E_m - E_n)}$$

$$\boxed{\frac{P_{n\to m}(t \gg)}{t} = \frac{2\pi}{\hbar} \delta(E_m - E_n) \left| \langle \psi_m | \hat{H}_1 | \psi_n \rangle \right|^2}$$

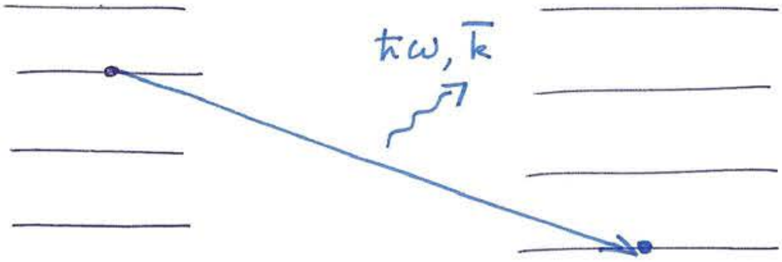
TRANSITION PROBABILITY PER UNIT OF TIME

ENERGY CONSERVATION

THIS IS KNOWN AS FERMI'S GOLDEN RULE

- EMISSION OF RADIATION BY AN EXCITED ATOM

↳



$|\Psi_i\rangle$ INITIAL

$|\Psi_f\rangle$ FINAL

$| \ldots n_{\bar{k}\sigma} \ldots \rangle$

$| \ldots \underbrace{n_{\bar{k}\sigma} + 1}_{} \ldots \rangle$

ONE EXTRA PHOTON WITH $\bar{k}\,\sigma$

$$|i\rangle \equiv \underset{\Uparrow}{|\Psi_i\rangle} \otimes | \ldots n_{\bar{k}\sigma} \ldots \rangle$$

MATTER STATE (e.g. e^- IN $2°$ EXCITED STATE OF ATOM)

$$|f\rangle \equiv |\Psi_f\rangle \otimes | \ldots n_{\bar{k}\sigma} + 1 \ldots \rangle$$

↳ TRANSITION PROBABILITY PER UNIT TIME $\frac{dP}{dt}$

$$\boxed{\frac{dP}{dt} = \frac{2\pi}{\hbar} \left| \mathcal{M}_{fi} \right|^2 \delta(E_i - E_f)}$$

(FERMI'S GOLDEN RULE)

↳ TRANSITION AMPLITUDE $\mathcal{M}_{fi}$

IN 1° ORDER PERTURBATION THEORY

$$\mathcal{M}_{fi} = \langle f \,|\, H'_{INT} \,|\, i \rangle$$

$$= \langle \Psi_f | \langle \ldots n_{\bar{k}\sigma} + 1 \ldots | \hat{H}'_{INT} | \ldots n_{\bar{k}\sigma} \ldots \rangle |\Psi_i\rangle$$

$$= -\frac{e}{mc} \sum_{\bar{k}'\sigma'} N_{k'} \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}'\sigma'}$$

$$\otimes \langle \ldots n_{\bar{k}\sigma} + 1 \ldots | \hat{a}_{\bar{k}'\sigma'} e^{i\bar{k}' \cdot \bar{x}} + \hat{a}^{+}_{\bar{k}'\sigma'} e^{-i\bar{k}' \cdot \bar{x}} | \ldots n_{\bar{k}\sigma} \ldots \rangle \otimes |\Psi_i\rangle$$

$$= -\frac{e}{mc} \left(\frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2} \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma}$$

$$\otimes \underbrace{\langle \ldots n_{\bar{k}\sigma} + 1 \ldots | \hat{a}^{+}_{\bar{k}\sigma} e^{-i\bar{k} \cdot \bar{x}} | \ldots n_{\bar{k}\sigma} \ldots \rangle}_{e^{-i\bar{k} \cdot \bar{x}} \sqrt{n_{\bar{k}\sigma} + 1}} \otimes |\Psi_i\rangle$$

$$\therefore \boxed{\mathcal{M}_{fi} = -\frac{e}{mc} \left(\frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2} \sqrt{n_{\bar{k}\sigma} + 1} \, \langle \Psi_f | \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} \, e^{-i\bar{k} \cdot \bar{x}} | \Psi_i \rangle}$$

$\therefore \quad \dfrac{dP}{dt} = \dfrac{2\pi}{\hbar} \, \delta(E_i - E_f) \, |\mathcal{M}_{fi}|^2$

$E_i = \underset{\substack{\uparrow \\ \text{MATTER}}}{E_{M_i}}$

$E_f = E_{M_f} + \hbar\omega$

$$\boxed{\begin{aligned} \frac{dP}{dt} = {} & \frac{2\pi}{\hbar} \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{(mc)^2} \cdot \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right) \cdot (n_{\bar{k}\sigma} + 1) \\ & \cdot \left| \langle \psi_f | \, \hat{\bar{p}} \cdot \bar{\varepsilon}_{\bar{k}\sigma} \, e^{-i\bar{k}\cdot\bar{x}} \, | \psi_i \rangle \right|^2 \\ & \cdot \delta(E_{M_i} - E_{M_f} - \hbar\omega_k) \end{aligned}}$$

$\Rightarrow \quad \dfrac{dP}{dt} \sim (n_{\bar{k}\sigma} + 1)$

$\therefore$ THE MORE PHOTONS AVAILABLE WITH $\bar{k}\sigma$ THE MORE LIKELY THE EMISSION OF ANOTHER ONE (BOSONS!)

↳ STIMULATED EMISSION (→ LASER)

$\therefore$ FOR $n_{\bar{k}\sigma} = 0$, NON-ZERO PROBABILITY

↳ SPONTANEOUS EMISSION.

- LIFETIME OF EXCITED STATE (SPONTANEOUS EMISSION)

$$n_{\bar{k}\sigma} = 0$$

$\downarrow$

τ

$$\hookrightarrow \left(\frac{1}{\tau_{i\to f}}\right) \equiv \frac{2\pi}{\hbar} \sum_{\bar{k}\sigma} |\langle f | H'_{INT} | i \rangle|^2 \, \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$|\bar{k}| = \frac{\omega_k}{c}$$

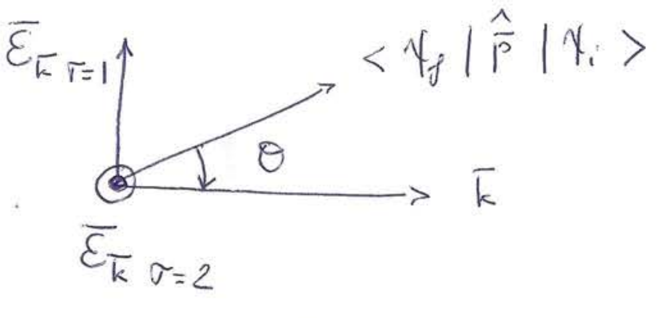
$\downarrow$

$$\frac{1}{\tau_{i\to f}} = \frac{2\pi}{\hbar} \cdot \frac{e^2}{4\pi} \cdot \frac{2\pi\hbar}{m^2 \quad L^3} \cdot \sum_{\bar{k}\sigma} \frac{1}{\omega_k} \cdot |\langle \psi_f | \hat{p} \cdot \bar{\varepsilon}_{\bar{k}\sigma} e^{-i\bar{k}\cdot\bar{x}} | \psi_i \rangle|^2 \cdot \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\downarrow \qquad \sum_{\bar{k}} \cdots = \frac{L^3}{(2\pi)^3} \int d^3\bar{k} \quad \cdots$$

$$\frac{1}{\tau_{i\to f}} = \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \, m^2} \int d^3\bar{k} \sum_{\sigma=1,2} \frac{1}{\omega_k} \, \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right) \cdot |\langle \psi_f | \hat{p} \cdot \bar{\varepsilon}_{\bar{k}\sigma} e^{-i\bar{k}\cdot\bar{x}} | \psi_i \rangle|^2$$

$\downarrow$ CHOOSE $\bar{\varepsilon}_{\bar{k}\,\sigma=1}$ IN PLANE $\bar{k}$, $\langle \psi_f | \hat{p} | \psi_i \rangle$



$$\bar{\varepsilon}_{\bar{k}\,\sigma=2} \cdot \langle \psi_f | \hat{\bar{p}}\, e^{-i\bar{k}.\bar{x}} | \psi_i \rangle = 0$$

↳ FOR <u>ATOMS</u> $\hbar\omega_k \approx 10\ eV$ (TYPICALLY)

$$\bar{k}.\bar{x} \approx \frac{2\pi}{\lambda}\, a_{BOHR}$$

$$= \frac{\hbar\omega_k}{\hbar c}\, a_{BOHR}$$

$\hbar c \approx 1970\ eV\ \text{Å}$
($1\ \text{Å} = 10^{-10}\ m$)

$$\approx \frac{10\ eV}{1970\ eV\ \text{Å}} \cdot \underbrace{a_{BOHR}}_{\approx 0.5\ \text{Å}}$$

$$\approx 2.5\ 10^{-3} \ll 1$$

LONG - WAVELENGTH APPROXIMATION

$$e^{-i\bar{k}.\bar{x}} = 1 - i\bar{k}.\bar{x} + \frac{1}{2}(i\bar{k}\,\bar{x})^2 + \ldots$$

$$\approx 1 \quad (\bar{k}.\bar{x} \ll 1)$$

$$\Downarrow$$

$$\bar{\varepsilon}_{\bar{k}\,\sigma=1} \cdot \langle \psi_f | \hat{\bar{p}}\, e^{-i\bar{k}.\bar{x}} | \psi_i \rangle$$

$$\approx \bar{\varepsilon}_{\bar{k}\,\sigma=1} \cdot \langle \psi_f | \hat{\bar{p}} | \psi_i \rangle$$

$$= \sin\theta\, |\langle \psi_f | \hat{\bar{p}} | \psi_i \rangle|$$

$$\hookrightarrow \quad \frac{1}{\tau_{i\to f}} \simeq \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \cdot m^2}$$

$$\cdot \int d^3\bar{k} \; \frac{1}{\omega_k} \; \delta\left(E_{M_i} - E_{M_f} - \hbar\omega_k\right)$$

$$\cdot \sin^2\theta \; \left|\langle \psi_f \,|\, \hat{\bar{p}} \,|\, \psi_i \rangle\right|^2$$

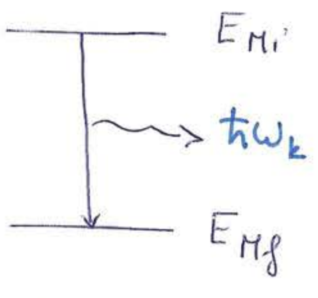
$\downarrow$ CHOOSE Z-AXIS ALONG $\langle \psi_f | \hat{\bar{p}} | \psi_i \rangle$

$$= \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi \cdot m^2} \cdot \left|\langle \psi_f \,|\, \hat{\bar{p}} \,|\, \psi_i \rangle\right|^2$$

$$\cdot (2\pi) \underbrace{\int_{-1}^{1} d\cos\theta \; (1-\cos^2\theta)}_{\frac{4}{3}} \cdot \int_0^\infty d|\bar{k}| \; \frac{|\bar{k}|^2}{|\bar{k}|c} \; \underbrace{\delta\left(E_{M_i} - E_{M_f} - \hbar c|\bar{k}|\right)}_{\frac{1}{\hbar c}\delta\left(|\bar{k}| - \frac{E_{M_i} - E_{M_f}}{\hbar c}\right)}$$

$$= \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{m^2} \; \frac{1}{\hbar c^2} \cdot \left(\frac{E_{M_i} - E_{M_f}}{\hbar c}\right) \cdot \frac{4}{3} \left|\langle \psi_f \,|\, \hat{\bar{p}} \,|\, \psi_i \rangle\right|^2$$

$$\boxed{\frac{1}{\tau_{i\to f}} = \frac{4}{3} \cdot \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{\hbar m^2 c^3} \left(\frac{E_{M_i} - E_{M_f}}{\hbar}\right) \cdot \left|\langle \psi_f \,|\, \hat{\bar{p}} \,|\, \psi_i \rangle\right|^2}$$



$$(E_{M_i} - E_{M_f}) \uparrow \;\Rightarrow\; \tau_{i\to f} \downarrow$$

$$\langle \psi_f | \hat{p} | \psi_i \rangle$$

$$= m \frac{d}{dt} \langle \psi_f | \hat{x} | \psi_i \rangle$$

$$= -\frac{im}{\hbar} \langle \psi_f | [\hat{x}, \hat{H}_M] | \psi_i \rangle$$

$$= m \left(-\frac{i}{\hbar}\right) (E_{Mi} - E_{Mf}) \langle \psi_f | \hat{x} | \psi_i \rangle$$

$$\therefore \quad \boxed{\frac{1}{\tau_{i \to f}} = \frac{4}{3} \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{\hbar c^3} \left(\frac{E_{Mi} - E_{Mf}}{\hbar}\right)^3 |\langle \psi_f | \hat{x} | \psi_i \rangle|^2}$$

$e\hat{x}$: ELECTRIC DIPOLE OPERATOR

$\therefore$ ELECTRIC DIPOLE APPROXIMATION.