Practice Exam
Theoretical Physics 5 : WS 2021/2022
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Exercise 1. (15 points) : Real scalar field in 1+1 dimensions

Consider the following Lagrangian for a real scalar field ϕ in 1+1 (one spatial and one time) dimensions

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2,$$

a) (*5 p.*) Construct the corresponding Hamiltonian and find the condition on classical field configurations $\phi_0(x)$ which minimizes the energy. *Hint:* Remember that the conjugate momentum π of a field ϕ is given by $\partial\mathcal{L}/\partial\dot{\phi}$ and the Hamiltonian by $H = \int \mathrm{d}x \left(\pi\dot{\phi} - \mathcal{L}\right)$.

b) (*5 p.*) Find the equations of motions (Euler Lagrange equations) for the field ϕ.

c) (*5 p.*) The static solution which interpolates between two vacuum states

$$\phi_0(x) = v \tanh\left(\sqrt{\frac{\lambda}{2}} v x\right)$$

is called the kink solution. Prove that the kink is indeed a solution of the equations of motion.

Exercise 2. (40 points) : Fermions interacting through Yukawa force

In fermion assemblies, the competition between kinetic and potential energies leads to the stability of the system. This exercise proposes to describe the ground state of such assemblies using the simplest semi-classical method.

a) (*5 p.*) Assuming that the average kinetic energy per particle is the same as for non-interacting non-relativistic fermions at zero temperature, show that it is given by

$$\langle T_A \rangle = \frac{T}{A} = \frac{3\hbar^2}{5m}\left(\frac{3\rho\pi^2}{s\sqrt{2}}\right)^{2/3},$$

where A is the total number of fermions, $\rho = A/V$ and m are the fermion density and mass, respectively, and s is the number of fermions per kinematical state, *e.g.* $s = 2$ for electrons $(e_\uparrow, e_\downarrow)$ and $s = 4$ for nucleons (protons and neutrons of both spin orientations: $p_\uparrow, p_\downarrow, n_\uparrow, n_\downarrow$). Give the total kinetic energy of an assembly homogeneously distributed inside a sphere of radius R as a function of A.

b) (*20 p.*) Assuming that the potential energy between a pair of fermions at a distance r_{12} is of Yukawa type

$$V_{12} = -g^2 \frac{e^{-\mu r_{12}}}{r_{12}},$$

show that the potential energy of the homogeneous spherical assembly is given by

$$V_Y = \frac{1}{2}\int_S d^3r_1\, d^3r_2\, \rho^2 V_{12} = \frac{9g^2A^2}{4Rx^2}\left[-\frac{2}{3} + \frac{1}{x} - \frac{1}{x^3} + \frac{e^{-2x}}{x}\left(1 + \frac{2}{x} + \frac{1}{x^2}\right)\right],$$

with $x \equiv \mu R$.

c) (*15 p.*) Write down the condition of (classical) extremization of the energy. Given that the minimum of energy corresponds here to the smallest value of R satisfying the condition, show that for large values of A the distribution shrinks for growing A. Discuss the dependence of the minimal radius with respect to m, g^2 and μ.

Question 3. (15 points): Dirac particle in a square-well potential

Consider a Dirac particle travelling along the positive z-direction and subject to the square-well potential

$$V(z) = \begin{cases} 0 & \text{when } z < -a/2 & \text{(region I)} \\ V_0 & \text{when } -a/2 \le z \le a/2 & \text{(region II)} \\ 0 & \text{when } a/2 < z & \text{(region III)}, \end{cases}$$

where the width $a > 0$ and the depth $V_0 < 0$. In regions I and III, the eigenvalue Dirac equation takes the form

$$\left(\vec{\alpha}\cdot\hat{\vec{p}}c + \beta m_0 c^2\right)\psi = E\psi,$$

while in region II, it has the form

$$\left(\vec{\alpha}\cdot\hat{\vec{p}}c + \beta m_0 c^2\right)\psi = (E - V_0)\psi.$$

Hints: Remember that the matrices $\vec{\alpha}$ and β in the standard (Dirac) representation are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

a) (*10 p.*) Write down the general solution $\psi(z)$ in the three regions with the spin in the z-direction.

b) (*5 p.*) Impose the continuity condition at $z = \pm a/2$ and write the corresponding equations.

Question 4 (30 points) : Cherenkov effect

An electron moving in free space cannot emit or absorb one photon. However, when this electron is moving in a dielectric medium this emission or absorption is allowed. In such a medium the speed of light c' is given by $c' = c/n$, with $n > 1$ the refraction index of the medium, leading to $c' < c$. In such medium, an electron can move faster than light, i.e. can have a speed $v > c'$.

a) (*10 p.*) Under such circumstances, a free moving electron of momentum $\hbar\vec{q}$ can emit a photon of momentum $\hbar\vec{k}$, if its velocity $v > c'$. This radiation is known as Cherenkov radiation and is emitted under a specific angle w.r.t. the direction of the incoming electron. Use energy-momentum conservation to calculate this (Mach) angle $\cos\theta = \vec{k}\cdot\vec{q}/|\vec{k}||\vec{q}|$, and show that for low photon energies it can be expressed only by c, n and v.

b) (*10 p.*) For a medium with a constant refraction index, the interaction Hamiltonian for emission/absorption of one photon is given by:

$$\hat{H}'_{int} = -\frac{e}{mc}\sum_{\vec{k}}\sum_{\sigma}\tilde{\mathcal{N}}_k\,\hat{\vec{p}}\cdot\vec{\varepsilon}_{\vec{k}\sigma}\left[a_{\vec{k}\sigma}\,e^{i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}\sigma}\,e^{-i\vec{k}\cdot\vec{x}}\right].$$

Here the only change to the free space situation is that the normalization constant is changed to $\tilde{\mathcal{N}}_k^2 = \frac{\hbar c^2}{2\omega_k L^3 n^2}$.

Use Fermi's golden rule to calculate the transition probability per unit of time for emission of such Cherenkov radiation by an electron.

Hint: The transition probability per unit of time for a transition from initial state i to final state f is given by

$$\frac{dP}{dt} = \frac{2\pi}{\hbar}\,|\langle f|H'_{int}|i\rangle|^2\,\delta(E_i - E_f).$$

c) (*10 p.*) While travelling through such medium the electron loses energy by radiating Cherenkov light. Use the previous result for dP/dt to show that the energy loss per unit of length

$$\frac{dW}{dx} = \frac{1}{v}\sum_{\vec{k},s}\hbar\omega_k\,\frac{dP}{dt},$$

is given by:

$$\frac{dW}{dx} = \frac{e^2}{4\pi c^2}\int_0^{\omega_{max}} d\omega_k\,\omega_k\left(1 - \frac{c^2}{n^2 v^2}\left(1 + \frac{\hbar\omega_k n^2}{2mc^2}\right)^2\right).$$

What determines ω_{max} ?