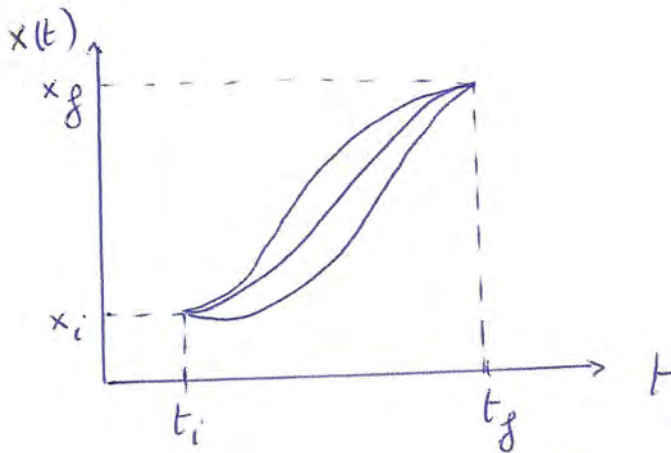


# PATH INTEGRALS IN QUANTUM MECHANICS

- 1) DERIVATION OF PATH INTEGRAL IN 1D
- 2) APPLICATION OF PATH INTEGRALS IN 1D
- 3) EXAMPLE IN 3D : AHARONOV-BOHM EFFECT
- 4) BROWNIAN MOTION AND WIENER PATH INTEGRAL

# 1) DERIVATION OF PATH INTEGRAL IN 1D

## • CLASSICAL ACTION



$$\text{ACTION } S[x(t)] = \int_{t_i}^{t_f} dt \, L(x(t), \dot{x}(t))$$

CONSIDER VARIATIONS OF  $S$  BY  $\delta x(t)$

SUCH THAT  $\delta x(t_i) = 0$  ,  $\delta x(t_f) = 0$

$x(t_i) = x_i$  ,  $x(t_f) = x_f$

$\Rightarrow$  VARIATIONAL PRINCIPLE (PRINCIPLE OF LEAST ACTION)

CLASSICAL PATH MINIMIZES  $S$

$$\text{i.e. } \underline{\underline{\delta S = 0}}$$

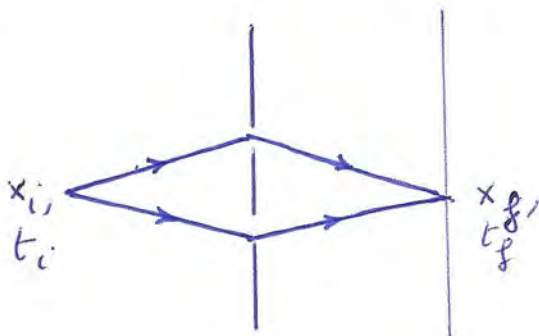
$\Downarrow$   
EULER-LAGRANGE EQ.

$$\underline{\underline{\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0}}$$

$\hookrightarrow$  SOLUTION GIVES CLASSICAL PATH

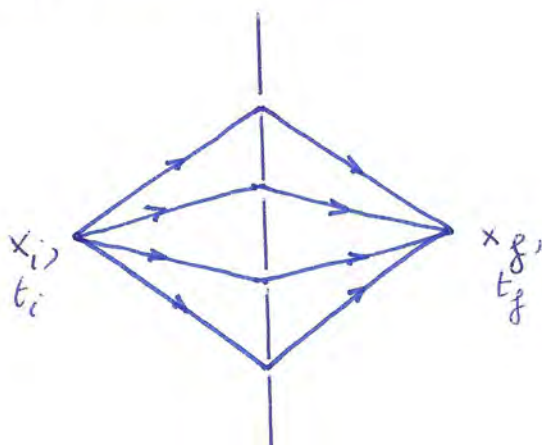
# • TRANSITION AMPLITUDE IN QUANTUM MECHANICS

↳ CONSIDER 2-SLIT EXP.



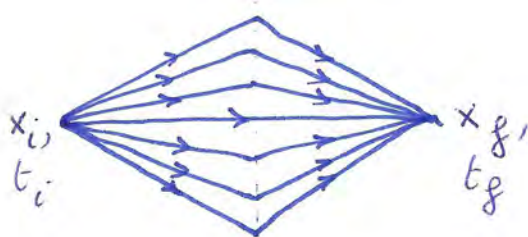
AT POSITION  $x_f$  & TIME  $t_f$   
 QM AMPLITUDE IS OBTAINED  
 AS SUM OF AMPLITUDE  
 FOR 2 PATHS  
 (INTERFERE)

↳ MAKE MORE SLITS



4 SLITS  
 QM AMPLITUDE AT  $x_f, t_f$   
 OBTAINED AS SUM  
 OVER 4 PATHS

↳ MAKE INFINITE # SLITS  $\rightarrow$  INTERMEDIATE SCREEN DISAPPEARS



QM AMPLITUDE AT  $x_f, t_f$   
 OBTAINED AS SUM  
 OVER ALL POSSIBLE PATHS

↳ CONSIDER POSITION OPERATOR IN SCHRÖDINGER PICTURE OF QM

$$\hat{X}_S |x\rangle_S = x |x\rangle_S$$

↑  
TIME-INDEP. OPERATOR

↑  
EIGENVALUE

↑  
TIME-INDEP. EIGENSTATES

IN SCHRÖDINGER PICTURE (S)

↳ IN HEISENBERG PICTURE (H) : OPERATOR & EIGENSTATES BECOME TIME DEPENDENT

$$\hat{X}_H(t) \equiv e^{\frac{i}{\hbar} \hat{H} t} \hat{X}_S e^{-\frac{i}{\hbar} \hat{H} t}$$

$\hat{H}$  HAMILTONIAN

$$|x, t\rangle \equiv e^{\frac{i}{\hbar} \hat{H} t} |x\rangle_S$$

↓

NOTE  $|x, t=0\rangle = |x\rangle_S = |x\rangle$  IN FOLLOWING

$$\hat{X}_H(t) |x, t\rangle = x |x, t\rangle$$

↳ TRANSITION AMPLITUDE (PROPAGATOR)

$$\langle x_f, t_f | x_i, t_i \rangle \equiv K(x_f, t_f; x_i, t_i)$$

PROBABILITY AMPLITUDE THAT SYSTEM WHICH IS IN EIGENSTATE  $|x_i, t_i\rangle$  AT TIME  $t_i$  WILL BE IN EIGENSTATE  $|x_f, t_f\rangle$  AT TIME  $t_f$

↳ INSERT COMPLETE SET OF STATES OF  $H$

$$\begin{aligned}
 & K(x_f, t_f; x_i, t_i) \\
 &= \langle x_f, t_f | x_i, t_i \rangle \\
 &= \langle x_f | e^{-\frac{i}{\hbar} \hat{H} (t_f - t_i)} | x_i \rangle \\
 &= \sum_m \sum_{m'} \langle x_f | \Psi_m \rangle \langle \Psi_m | e^{-\frac{i}{\hbar} \hat{H} (t_f - t_i)} | \Psi_{m'} \rangle \langle \Psi_{m'} | x_i \rangle
 \end{aligned}$$

$|\Psi_m\rangle$  : EIGENSTATES OF  $\hat{H}$

$$\hat{H} |\Psi_m\rangle = E_m |\Psi_m\rangle$$

$$\langle \Psi_m | \Psi_{m'} \rangle = \delta_{mm'}$$

$$= \sum_m \underbrace{\langle x_f | \Psi_m \rangle}_{\Psi_m(x_f)} e^{-\frac{i}{\hbar} E_m (t_f - t_i)} \underbrace{\langle \Psi_m | x_i \rangle}_{\Psi_m^*(x_i)}$$

$$= \sum_m e^{-\frac{i}{\hbar} E_m (t_f - t_i)} \Psi_m(x_f) \Psi_m^*(x_i)$$

FOURIER ANALYZING GIVES EIGENVALUES  $E_m$  OF  $\hat{H}$

FOURIER COEFFICIENTS EIGENSTATES  $|\Psi_m\rangle$

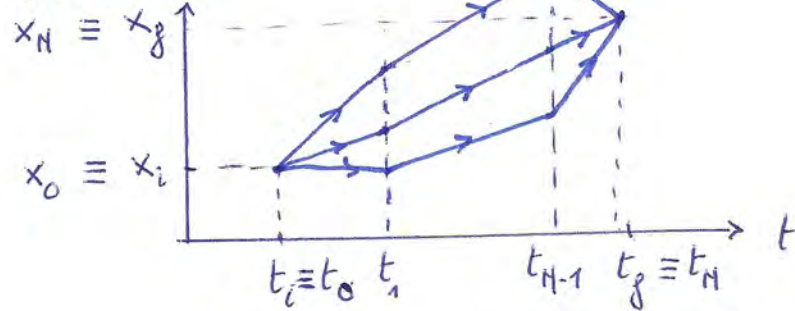
PROPAGATOR CONTAINS ALL DYNAMICAL INFORMATION ON QUANTUM SYSTEM

# • TRANSITION AMPLITUDE AS PATH INTEGRAL

R.P. FEYNMAN (1948)

↳ BREAK UP TIME INTERVAL  $t_f - t_i$  INTO  $N$  INFINITESIMAL

INTERVALS  $\Delta t$



$$t_k = t_i + k \Delta t$$

+ USE COMPLETENESS OF EIGENSTATES  
AT INTERMEDIATE  $t_k$

$$\int_{-\infty}^{+\infty} dx_k |x_k, t_k\rangle \langle x_k, t_k| = \mathbb{1}$$

↳  $K(x_f, t_f; x_i, t_i)$

$$= \prod_{k=1}^{N-1} \int dx_k \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \dots$$

$$\dots \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1, t_1 | x_0, t_0 \rangle$$

↳ FOR INFINITESIMAL INTERVAL

$$\langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \langle x_{k+1} | e^{-\frac{i}{\hbar} \hat{H}(t_{k+1} - t_k)} | x_k \rangle$$

$$= \langle x_{k+1} | e^{-\frac{i}{\hbar} \hat{H}(\Delta t)} | x_k \rangle$$

INSERT COMPLETE SET OF MOMENTUM  
EIGENSTATES

$$\hat{P} |P_k\rangle = P_k |P_k\rangle$$

$$\langle P_k | P_{k'} \rangle = \delta(P_{k'} - P_k)$$

$$\int_{-\infty}^{+\infty} dP_k |P_k\rangle \langle P_k| = \mathbb{1}$$

$$\therefore \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \int dP_k \langle x_{k+1} | P_k \rangle \langle P_k | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | x_k \rangle$$

$$\downarrow \quad \hat{H} \quad \text{DEPENDS ON } \hat{x} \text{ \& } \hat{P}$$

$$= \int dP_k \langle x_{k+1} | P_k \rangle \langle P_k | x_k \rangle e^{-\frac{i}{\hbar} H(x_k, P_k) \Delta t}$$

NOTE: THIS IS THE  
HAMILTONIAN FUNCTION  
NOT OPERATOR ANY MORE

$$\text{e.g. } H(x, p) = \frac{p^2}{2m} + V(x)$$

↳ USE

$$\int dP_k \langle x_{k+1} | P_k \rangle \langle P_k | x_k \rangle = \langle x_{k+1} | x_k \rangle$$

$$= \delta(x_{k+1} - x_k)$$

$$= \frac{1}{2\pi\hbar} \int dP_k e^{+\frac{i}{\hbar} P_k (x_{k+1} - x_k)}$$

$$\langle x_{k+1} | P_k \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} P_k x_{k+1}}$$

$$\langle P_k | x_k \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} P_k x_k}$$

$$\hookrightarrow \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \int \frac{dP_k}{2\pi\hbar} e^{\frac{i}{\hbar} [P_k (x_{k+1} - x_k) - H(x_k, P_k) \cdot \Delta t]}$$

↓ FOR  $\Delta t$  INFINITESIMAL

$$x_{k+1} - x_k = \dot{x}_k (\Delta t)$$

$$\langle x_{k+1}, t_{k+1} | x_k, t_k \rangle = \int \frac{dP_k}{2\pi\hbar} e^{\frac{i}{\hbar} [P_k \dot{x}_k - H(x_k, P_k)] \Delta t}$$



## ↳ FINITE TRANSITION AMPLITUDE

$$\begin{aligned}
 & K(x_f, t_f; x_i, t_i) \\
 &= \prod_{k=1}^{N-1} \int dq_k \prod_{k=0}^{N-1} \int \frac{dP_k}{2\pi\hbar} \\
 & \cdot \exp \left\{ \frac{i}{\hbar} \sum_{k=0}^{N-1} \underbrace{[P_k \dot{x}_k - H(x_k, P_k)] \Delta t}_{\Delta t \rightarrow 0} \right\} \\
 & \rightarrow \int_{t_i}^{t_f} dt [P \dot{x} - H(x, P)]
 \end{aligned}$$

## ↳ PATH INTEGRAL

$$\begin{aligned}
 & K(x_f, t_f; x_i, t_i) \\
 &= \int \mathcal{D}x(t) \mathcal{D}p(t) \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt [P \dot{x} - H(x, P)] \right\}
 \end{aligned}$$

WITH PATH INTEGRAL 'MEASURES'

$$\mathcal{D}x(t) \equiv \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} dx(t_k)$$

$$\mathcal{D}p(t) \equiv \lim_{N \rightarrow \infty} \prod_{k=0}^{N-1} \frac{dp(t_k)}{2\pi\hbar}$$

WITH  $x(t_i) = x_i$  AND  $x(t_f) = x_f$

NOTE : IN EXPONENTIAL WE HAVE CLASSICAL  
ACTION FUNCTION OF SYSTEM  
IN TERMS OF HAMILTONIAN

$$S = \int_{t_i}^{t_f} dt [p \dot{x} - H(x, p)]$$

∴ PATH INTEGRAL IS

- FUNCTIONAL INTEGRAL OVER ALL POSSIBLE TRAJECTORIES IN PHASE SPACE OF SYSTEM
- WEIGHTED BY  $e^{\frac{i}{\hbar} S}$  WITH  
S THE HAMILTONIAN ACTION

# PATH INTEGRAL IN TERMS OF LAGRANGIAN ACTION

FOR  $H(x, p) = \frac{p^2}{2m} + V(x)$

WE CAN PERFORM THE  $p_k$  INTEGRATIONS (FORMALLY)

$$\hookrightarrow \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar} [p_k \dot{x}_k - H(x_k, p_k)] \Delta t}$$

$$= e^{-\frac{i}{\hbar} V(x_k) \Delta t} \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar} \left[ -\frac{p_k^2}{2m} + p_k \dot{x}_k \right] \Delta t}$$

$$= e^{-\frac{i}{\hbar} V(x_k) \Delta t} e^{+\frac{i}{\hbar} \frac{1}{2} m \dot{x}_k^2} \int \frac{dp_k}{2\pi\hbar} e^{-\frac{i \Delta t}{\hbar 2m} (p_k - m \dot{x}_k)^2}$$

ANALYTICAL CONTINUATION TO IMAG. TIME

$$\Delta \tau = i \Delta t \quad \text{REAL}$$

+ GAUSSIAN INTEGRAL

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$= e^{+\frac{i}{\hbar} \left[ \frac{1}{2} m \dot{x}_k^2 - V(x_k) \right] \Delta t} \cdot \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2}$$

$$= \underline{\underline{\left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} L(x_k, \dot{x}_k) \Delta t}}}$$

$$\begin{aligned}
 & \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle \\
 &= \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} L(x_k, \dot{x}_k) \Delta t}
 \end{aligned}$$

WITH LAGRANGIAN FUNCTION

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$$

↳ FINITE TRANSITION AMPLITUDE

$$\begin{aligned}
 & K(x_f, t_f; x_i, t_i) \\
 &= \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \int_{-\infty}^{+\infty} \prod_{k=1}^{N-1} dx_k e^{\frac{i}{\hbar} \sum_{k=0}^{N-1} L(x_k, \dot{x}_k) \Delta t}
 \end{aligned}$$

⇓

$$\begin{aligned}
 & K(x_f, t_f; x_i, t_i) \\
 &= \int \tilde{\mathcal{D}}_x(t) \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x, \dot{x}) \right\} \\
 &= \int \tilde{\mathcal{D}}_x(t) \exp \left\{ \frac{i}{\hbar} S \right\}
 \end{aligned}$$

WITH

$$\tilde{\mathcal{D}}_x(t) \equiv \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \prod_{k=1}^{N-1} dx(t_k)$$

## PATH INTEGRAL

↳ SUM OVER ALL PATHS  $x(t)$

↳ EACH PATH WEIGHTED WITH  $e^{\frac{i}{\hbar} S}$

↳ CLASSICAL LIMIT  $\hbar \rightarrow 0$

$e^{\frac{i}{\hbar} S}$  OSCILLATES RAPIDLY WHEN  $\hbar \rightarrow 0$

ONLY PATH WHICH MAKES ACTION  $S$  STATIONARY  
CONTRIBUTES TO PATH INTEGRAL

$$\delta S[x] = 0$$



$$x(t) = x_{cl}(t)$$

CLASSICAL PATH  
(SOLUTION OF EULER-LAGRANGE EQ.)

• EQUIVALENCE WITH SCHRÖDINGER EQUATION

↳ WAVEFUNCTION

$$\begin{aligned} \Psi(x_f, t_f) &= \langle x_f, t_f | \Psi(t=0) \rangle \\ &= \langle x_f | \underbrace{e^{-\frac{i}{\hbar} \hat{H} t_f}}_{| \Psi(t_f) \rangle} | \Psi(t=0) \rangle \\ &= \langle x_f | \Psi(t_f) \rangle \end{aligned}$$

↳ CONNECTION WITH TRANSITION AMPLITUDE

$$\begin{aligned} \Psi(x_f, t_f) &= \int dx_i \langle x_f, t_f | x_i, t_i \rangle \langle x_i, t_i | \Psi(t=0) \rangle \\ &= \int dx_i K(x_f, t_f; x_i, t_i) \Psi(x_i, t_i) \end{aligned}$$

↳ CONSIDER INFINITESIMAL TIME STEP

$$\begin{aligned} t_i &= t \\ t_f &= t + \Delta t \quad (\Delta t \rightarrow 0) \end{aligned}$$

$$K(x_f, t + \Delta t; x_i, t) = \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} L\left(\frac{x_i + x_f}{2}, \frac{x_f - x_i}{\Delta t}\right) \Delta t}$$

$$\Psi(x_f, t + \Delta t) = \int dx_i K(x_f, t + \Delta t; x_i, t) \Psi(x_i, t)$$

↳ CONSIDER PARTICLE MOVING IN 1D

$$L = \frac{1}{2} m \dot{x}^2 - V(x, t)$$

$$\rightsquigarrow K(x_f, t + \Delta t; x_i, t) = \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} \frac{m}{2\Delta t} (x_f - x_i)^2} e^{-\frac{i}{\hbar} \Delta t V\left(\frac{x_i + x_f}{2}, t\right)}$$

$x_f \equiv x$   
 $x_i \equiv y$

$$\rightsquigarrow \Psi(x, t + \Delta t) = \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} \int_{-\infty}^{+\infty} dy e^{-\frac{m}{2\hbar i \Delta t} (x-y)^2} e^{-\frac{i}{\hbar} \Delta t V\left(\frac{x+y}{2}, t\right)} \Psi(y, t)$$

ONLY REGION  $y \approx x$  CONTRIBUTES MAINLY TO INTEGRAL  
(ONLY REGION AROUND  $\eta \approx 0$ )

$$y = x + \eta$$

CHANGE INTEGRATION VARIABLE

$$\Psi(x, t + \Delta t) = \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} \int_{-\infty}^{+\infty} d\eta e^{-\frac{m \eta^2}{2\hbar i \Delta t}} e^{-\frac{i}{\hbar} \Delta t V\left(x + \frac{\eta}{2}, t\right)} \Psi(x + \eta, t)$$

EXPAND IN  $\Delta t$  AND KEEP ONLY LINEAR TERMS IN  $\Delta t$  & UP TO QUADRATIC TERMS IN  $\eta$

$$\begin{aligned} & \Psi(x, t) + \Delta t \frac{\partial \Psi}{\partial t} \\ &= \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} \int_{-\infty}^{+\infty} d\eta e^{-\frac{m}{2\hbar i \Delta t} \eta^2} \left[ 1 - \frac{i}{\hbar} \Delta t V(x, t) + \dots \right] \\ & \quad \cdot \left[ \Psi(x, t) + \eta \frac{\partial \Psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \Psi}{\partial x^2} + \dots \right] \end{aligned}$$

$$\rightsquigarrow \left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta e^{-a\eta^2} = 1 \quad \left(a = \frac{m}{2\hbar i \Delta t}\right)$$

$$\left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta \eta e^{-a\eta^2} = 0$$

$$\begin{aligned} \left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta \eta^2 e^{-a\eta^2} &= \left(\frac{a}{\pi}\right)^{1/2} \left(-\frac{1}{2a}\right) \int_{-\infty}^{+\infty} d(e^{-a\eta^2}) \eta \\ &= \left(\frac{a}{\pi}\right)^{1/2} \frac{1}{2a} \int_{-\infty}^{+\infty} d\eta e^{-a\eta^2} \\ &= \frac{1}{2a} \end{aligned}$$

$$\begin{aligned} \rightsquigarrow \psi(x,t) + \Delta t \frac{\partial \psi}{\partial t} \\ = \psi(x,t) - \frac{i}{\hbar} \Delta t V(x,t) \psi(x,t) \\ + \left(\frac{\hbar i \Delta t}{m}\right) \cdot \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \end{aligned}$$

IDENTIFY TERM  $O(\Delta t)$

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t) \psi \right\}$$

↓  
SCHRÖDINGER EQ !

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t) \psi = i\hbar \frac{\partial \psi}{\partial t}$$



## 2) APPLICATION OF PATH INTEGRALS IN 1D

- FREE PARTICLE ( $V=0$ )

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2$$

$$\rightsquigarrow K_0(x_f, t_f; x_i, t_i) = \int_{\tilde{\mathcal{D}} x(t)} e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \left( \frac{1}{2} m \dot{x}^2 \right)}$$

$K_0$

0 DENOTES  
FREE PARTICLE

$$= \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \int_{-\infty}^{+\infty} \prod_{k=1}^{N-1} dx_k e^{\frac{i}{\hbar} \Delta t \sum_{k=0}^{N-1} \frac{1}{2} m \frac{(x_{k+1} - x_k)^2}{(\Delta t)^2}}$$

$$\text{WITH } \begin{cases} x_0 \equiv x_i \\ x_N \equiv x_f \end{cases}$$

$\rightsquigarrow$  PERFORM FIRST  $\int dx_1$

$$\left( \frac{m}{2\pi\hbar i \Delta t} \right)^{2/2} \int dx_1 e^{-\left( \frac{m}{2\hbar i \Delta t} \right) \left\{ (x_2 - x_1)^2 + (x_1 - x_0)^2 \right\}}$$

$$\begin{aligned} & (x_2 - x_1)^2 + (x_1 - x_0)^2 \\ &= 2x_1^2 - 2(x_0 + x_2)x_1 + x_0^2 + x_2^2 \\ &= 2 \left( x_1 - \frac{1}{2}(x_0 + x_2) \right)^2 + \frac{1}{2}(x_0 - x_2)^2 \end{aligned}$$

$$= \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} \cdot \frac{1}{\sqrt{2}} e^{-\frac{1}{2} \left( \frac{m}{2\hbar i \Delta t} \right) (x_0 - x_2)^2}$$

∴ AFTER  $\int dx_1$  INTEGRATION

$$\left( \frac{m}{2\pi\hbar i \underline{2\Delta t}} \right)^{1/2} e^{-\left( \frac{m}{2\hbar i \underline{2\Delta t}} \right) (x_2 - x_0)^2}$$

→ MULTIPLY WITH  $\left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2}$  AND  $\int dx_2 e^{-\frac{m}{2\hbar i \Delta t} (x_3 - x_2)^2}$

$$\left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} \cdot \left( \frac{m}{2\pi\hbar i \underline{2\Delta t}} \right)^{1/2}$$

$$\cdot \int dx_2 e^{-\frac{m}{2\hbar i \Delta t} \left\{ (x_3 - x_2)^2 + \frac{1}{2} (x_2 - x_0)^2 \right\}}$$

$$\begin{aligned} & \left( x_3 - x_2 \right)^2 + \frac{1}{2} \left( x_2 - x_0 \right)^2 \\ &= \frac{3}{2} x_2^2 - (2x_3 + x_0) x_2 + x_3^2 + \frac{1}{2} x_0^2 \\ &= \frac{3}{2} \left( x_2 - \frac{1}{3} (2x_3 + x_0) \right)^2 \\ &+ x_3^2 + \frac{1}{2} x_0^2 - \frac{1}{6} (4x_3^2 + 4x_0 x_3 + x_0^2) \\ & \qquad \qquad \qquad \frac{1}{3} (x_3 - x_0)^2 \end{aligned}$$

$$= \left( \frac{m}{2\pi\hbar i \underline{3\Delta t}} \right)^{1/2} e^{-\left( \frac{m}{2\hbar i \underline{3\Delta t}} \right) (x_3 - x_0)^2}$$

→ AFTER  $N-1$  INTEGRATIONS.

$$\left( \frac{m}{2\pi\hbar i N\Delta t} \right)^{1/2} e^{-\left( \frac{m}{2\hbar i N\Delta t} \right) \left( \begin{matrix} x_f & - & x_i \\ \parallel & & \parallel \\ x_N & & x_0 \end{matrix} \right)^2}$$

$$\begin{aligned} \circ \circ \quad K_0(x_f, t_f; x_i, t_i) \\ = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi\hbar i N \Delta t} \right)^{1/2} e^{-\frac{m}{2\hbar i N \Delta t} (x_f - x_i)^2} \end{aligned}$$

$$\downarrow \quad N \Delta t = t_f - t_i$$

$$\begin{aligned} K_0(x_f, t_f; x_i, t_i) \\ = \left( \frac{m}{2\pi\hbar i (t_f - t_i)} \right)^{1/2} \exp \left\{ \frac{im}{2\hbar} \frac{(x_f - x_i)^2}{(t_f - t_i)} \right\} \end{aligned}$$

→ NOTE : FOR CLASSICAL FREE PARTICLE

$$x_{cl}(t) = x_i + \left( \frac{x_f - x_i}{t_f - t_i} \right) t$$

$$\dot{x}_{cl} = \frac{x_f - x_i}{t_f - t_i} \quad \text{CONSTANT}$$

$$S_{cl} = \int_{t_i}^{t_f} dt \left( \frac{1}{2} m \dot{x}_{cl}^2 \right)$$

$$S_{cl} = \frac{m}{2} \frac{(x_f - x_i)^2}{(t_f - t_i)}$$

FOR QUANTUM FREE PARTICLE

$$K_0(x_f, t_f; x_i, t_i) \sim e^{\frac{i}{\hbar} S_{cl}}$$

→ MOMENTUM OF FREE PARTICLE

TAKE FREE PARTICLE INITIALLY AT  $x_i = 0, t_i = 0$

• CLASSICAL  $S_{cl} = \frac{m}{2} \frac{x_f^2}{t_f}$

$\frac{\partial S_{cl}}{\partial x_f} = m \left( \frac{x_f}{t_f} \right) = P$   
 ↑ VELOCITY  $v$       ↑ MOMENTUM

• QM : PROBABILITY AMPLITUDE FOR PARTICLE TO BE FOUND AT  $x_f, t_f$

$K_0(x_f, t_f; 0, 0)$   
 $= \left( \frac{m}{2\pi\hbar i t_f} \right)^{1/2} e^{i \frac{m}{2\hbar} \frac{x_f^2}{t_f}}$

↳ FOR FIXED  $t_f$  AS  $x_f$  VARIES  
 OSCILLATES

⇓  
 PARTICLE BEHAVES AS WAVE

WAVELENGTH : COMPUTED FROM PERIODICITY CONDITION

$2\pi = \frac{m}{2\hbar t_f} [(x_f + \lambda)^2 - x_f^2]$

↓  $x_f \gg \lambda$

$2\pi = \frac{m}{\hbar} \left( \frac{x_f}{t_f} \right) \lambda \Rightarrow \lambda = \frac{h}{P}$   
 DE BROGLIE

WITH  $P = m \frac{x_f}{t_f}$   
 ↑  
 CLASSICAL MOMENTUM

## ~> ENERGY OF FREE PARTICLE

- CLASSICAL

$$- \frac{\partial S_{cl}}{\partial t_f} = \frac{1}{2} m \frac{x_f^2}{t_f^2} = \frac{1}{2} m v^2 = E$$

- QM

FOR FIXED  $x_f$

$K_0$  OSCILLATES AS  $t_f$  VARIES

FREQUENCY  $\omega = \frac{2\pi}{T}$       T: PERIOD

$$2\pi = \frac{m}{2\hbar} x_f^2 \left[ \frac{1}{t_f} - \frac{1}{t_f + T} \right]$$

$$= \frac{m x_f^2}{2\hbar} \frac{T}{(t_f + T) t_f}$$

$$\downarrow \quad t_f \gg T$$

$$\frac{2\pi}{T} \approx \frac{m}{2\hbar} \frac{x_f^2}{t_f} = \frac{E}{\hbar}$$

$$\boxed{E = \hbar \omega}$$

CLASSICAL ENERGY  $E = \frac{1}{2} m \frac{x_f^2}{t_f^2}$

## HARMONIC OSCILLATOR

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$\rightsquigarrow K(x, t; x_i, 0) = \int_{x_i}^{x(t)=x} \tilde{\mathcal{D}} x(t') e^{\frac{i}{\hbar} S[x(t)]}$$

$$(CHOOSE t_i = 0) \quad x(0) = x_i$$

$$S[x(t')] = \int_0^t dt' \left[ \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right]$$

## CLASSICAL PATH

$x_{cl}(t)$  IS SOLUTION OF E.L EQ.

$$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0$$

$$-m\omega^2 x - m\ddot{x} = 0$$

$$\ddot{x} = -\omega^2 x$$

$$\downarrow$$

$$x_{cl}(t') = A \cos \omega t' + B \sin \omega t'$$

CONSTRAINTS  $x_{cl}(0) = x_i \rightarrow A = x_i$

$$x_{cl}(t) = x \rightarrow B = \frac{x - x_i \cos \omega t}{\sin \omega t}$$

$$x_{cl}(t') = x_i \cos \omega t' + \frac{(x - x_i \cos \omega t)}{\sin \omega t} \sin \omega t'$$

→ CLASSICAL ACTION

$$\begin{aligned}
 S_{cl} \equiv S[x_{cl}] &= \frac{m\omega^2}{2} \int_0^t dt' \left\{ \left[ -A \sin \omega t' + B \cos \omega t' \right]^2 \right. \\
 &\quad \left. - \left[ A \cos \omega t' + B \sin \omega t' \right]^2 \right\} \\
 &= \frac{m\omega^2}{2} \int_0^t dt' \left\{ + \cos 2\omega t' (B^2 - A^2) \right. \\
 &\quad \left. - \sin 2\omega t' 2AB \right\} \\
 &= \frac{m\omega}{4} \left\{ + (B^2 - A^2) \sin 2\omega t' \Big|_0^t \right. \\
 &\quad \left. + 2AB \cos 2\omega t' \Big|_0^t \right\} \\
 &= \frac{m\omega}{4} \left\{ (-A^2 + B^2) \sin 2\omega t + 2AB (\cos 2\omega t - 1) \right\}
 \end{aligned}$$

$$S[x_{cl}] = \frac{m\omega}{2 \sin \omega t} \left\{ (x^2 + x_i^2) \cos \omega t - 2x x_i \right\}$$

→ QUANTUM ACTION

$$x(t') = x_{cl}(t') + Y(t')$$

↑  
DEVIATION FROM CLASSICAL PATH

$$Y(0) = 0$$

$$Y(t) = 0$$

$$S[x(t')] = \frac{m}{2} \int_0^t dt' \left[ (\dot{x}_{cl} + \dot{Y})^2 - \omega^2 (x_{cl} + Y)^2 \right]$$

$$= \frac{m}{2} \int_0^t dt' \left\{ \dot{x}_{cl}^2 - \omega^2 x_{cl}^2 + 2\dot{x}_{cl}\dot{Y} - 2\omega^2 x_{cl}Y + \dot{Y}^2 - \omega^2 Y^2 \right\}$$

NOTE :  $\int_0^t dt' 2\dot{x}_{cl}\dot{Y} = 2\dot{x}_{cl}Y \Big|_0^t - \int_0^t dt' 2Y\ddot{x}_{cl}$

BECAUSE  $Y(t) = Y(0) = 0$

$$\hookrightarrow \int_0^t dt' (2\dot{x}_{cl}\dot{Y} - 2\omega^2 x_{cl}Y)$$

$$= -2 \int_0^t dt' Y (\ddot{x}_{cl} + \omega^2 x_{cl})$$

$$= 0$$

↙ CLASSICAL EQUATION OF MOTION



$$\circ \circ \quad \left\| \begin{aligned} S[x(t')] &= S[x_{cl}] \\ &+ \frac{m}{2} \int_0^t dt' [\dot{y}^2 - \omega^2 y^2] \end{aligned} \right.$$

NOTE THAT 2<sup>nd</sup> TERM DOES NOT  
DEPEND ON  $x$  OR  $x_i$

WE CAN THEREFORE WRITE:

$$K(x, t; x_i, 0) = K(0, t; 0, 0) e^{\frac{i}{\hbar} S[x_{cl}]}$$

→ SHOW THAT

$$K(0, t; 0, 0) = \left( \frac{m\omega}{2\pi i\hbar \sin\omega t} \right)^{1/2}$$

$$\rightarrow K(x, t; x_i, 0)$$

$$= \left( \frac{m\omega}{2\pi i\hbar \sin\omega t} \right)^{1/2} e^{\frac{i}{\hbar} S[x_{cl}]}$$

→ NOTE : IN GENERAL → WHEN  $S$  CAN BE EXPRESSED  
THROUGH A QUADRATIC FORM

$$K \sim e^{\frac{i}{\hbar} S[x_{cl}]}$$

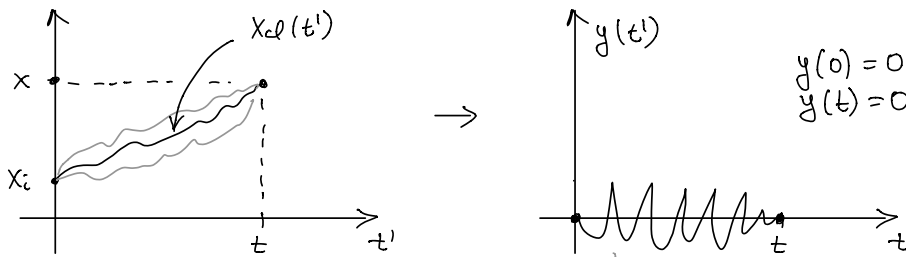
Let's show that

$$K(0, t, 0, 0) = \left( \frac{m \omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2}$$

Due to our replacement

$$X(t') = X_{cl}(t') + y(t')$$

deviation from classical path



$$K(0, t, 0, 0) = \int_0^0 \mathcal{D}y(t) \exp\left(\frac{i}{\hbar} \frac{m}{2} \int_0^t dt' (\dot{y}^2 - \omega^2 y)\right)$$

such path can be written as a Fourier sine series with a fundamental period of  $t$

$$y(t') = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi t'}{t}\right)$$

and it is possible to specify a path through the coefficients  $A_n$  instead of the function values  $y(t_k)$ .

The jacobian of this transformation  $J$  doesn't depend on  $\omega$ .

General: all the prefactors that do not depend on  $\omega$  we will recover from  $\omega=0$  limit which corresponds to free particle

$$K(0, t, 0, 0) \xrightarrow{\omega=0} \left( \frac{m}{2\pi i \hbar t} \right)^{1/2}$$

Plug in  $y(t')$  into  $\exp(-)$ :

$$\begin{aligned} 1) \quad \frac{m}{2} \int_0^t dt' \dot{y}^2 &= \frac{m}{2} \int_0^t dt' \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{t}\right) \cos\left(\frac{n\pi t'}{t}\right) \sum_{m=1}^{\infty} a_m \left(\frac{m\pi}{t}\right) \cos\left(\frac{m\pi t'}{t}\right) \\ &= \frac{m}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{n\pi}{t}\right) \left(\frac{m\pi}{t}\right) \int_0^t dt' \cos\left(\frac{n\pi t'}{t}\right) \cos\left(\frac{m\pi t'}{t}\right) \end{aligned}$$

using the relation

$$\cos(a) \cos(b) = \frac{1}{2} (\cos(a-b) + \cos(a+b))$$

$$= \frac{m}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{n\pi}{t}\right) \left(\frac{m\pi}{t}\right) \frac{t}{2\pi} \left( \underbrace{\frac{\sin(\pi(n-m))}{n-m}}_{n \neq m} + \underbrace{\frac{\sin(\pi(n+m))}{n+m}}_{n \neq -m} \right)$$

$$= \frac{m}{2} \frac{t}{2} \sum_{n=1}^{\infty} \left(\frac{n\pi}{t}\right)^2 a_n^2 \quad \begin{array}{ll} n \neq m & 0 \\ n = m & \pi \end{array} \quad \begin{array}{ll} n \neq -m & 0 \\ n = -m & \pi \end{array}$$

not possible  
 $n, m = \{1, \infty\}$

Similarly

using the relation

$$\sin(a) \sin(b) = \frac{1}{2} (\cos(a-b) - \cos(a+b))$$

$$2) \quad \frac{m\omega^2}{2} \int_0^t dt' y^2 = \frac{m\omega^2}{2} \frac{t}{2} \sum_{n=1}^{\infty} a_n^2$$

On the assumption that  $[0, t]$  region is divided into discrete steps, there is only a finite number  $N$  of coefficients  $a_n$

$$k(0, t, 0, 0) = (J \dots) \int_{-p}^{+p} da_1 \int_{-p}^{+p} da_2 \dots \int_{-p}^{+p} da_N \exp \left\{ \frac{i m t}{2} \sum_{n=1}^{\infty} \left( \left(\frac{n\pi}{t}\right)^2 - \omega^2 \right) a_n^2 \right\}$$

some factors that do not depend on  $\omega$

Since the exp. can be separated into factors, the integral over each coefficients  $a_n$  can be done separately

$$\int_{-\infty}^{+\infty} da_n \exp \left\{ \frac{i m}{2 \hbar} \frac{t}{z} \left( \frac{\pi^2 \hbar^2}{t^2} - \omega^2 \right) a_n^2 \right\} = (\dots) \cdot \left( \frac{\pi^2 \hbar^2}{t^2} - \omega^2 \right)^{-1/2} =$$

↑ doesn't depend on  $\omega$

Gaussian integral  $\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

$$= (\dots) \left( 1 - \frac{\omega^2 t^2}{\pi^2 \hbar^2} \right)^{-1/2}$$

Therefore

$$k(0, t, 0, 0) = (\dots) \prod_{n=1}^N \left( 1 - \frac{\omega^2 t^2}{\pi^2 \hbar^2} \right)^{-1/2} = (\dots) \left( \frac{\sin \omega t}{\omega t} \right)^{-1/2}$$

$N \rightarrow \infty$

Since

$$k(0, t, 0, 0) \xrightarrow{\omega=0} \left( \frac{m}{2 \pi \hbar i t} \right)^{1/2}$$

$$\Rightarrow \boxed{k(0, t, 0, 0) = \left( \frac{m \omega}{2 \pi i \hbar \sin(\omega t)} \right)^{1/2}} \quad \neq$$

• PROJECTION OF THE GROUND STATE:

FEYNMAN-KAC FORMULA

$$\begin{aligned}
 &\rightsquigarrow K(x, t; x', 0) \\
 &= \langle x, t | x', 0 \rangle \\
 &= \langle x | e^{-\frac{i}{\hbar} \hat{H} t} | x' \rangle \\
 &= \sum_m \langle x | \psi_m \rangle \langle \psi_m | x' \rangle e^{-\frac{i}{\hbar} E_m t}
 \end{aligned}$$

$\rightsquigarrow$  FOR  $x = x'$

$$\begin{aligned}
 &K(x, t; x, 0) \\
 &= \sum_m |\psi_m(x)|^2 e^{-\frac{i}{\hbar} E_m t}
 \end{aligned}$$

$\Downarrow$

$$\begin{aligned}
 &\int dx K(x, t; x, 0) \\
 &= \int dx \langle x | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle \\
 &= \text{Tr} e^{-\frac{i}{\hbar} \hat{H} t} \\
 &= \sum_m \underbrace{\int dx |\psi_m(x)|^2}_1 e^{-\frac{i}{\hbar} E_m t}
 \end{aligned}$$

$$\int dx K(x, t; x, 0) = \sum_n e^{-\frac{i}{\hbar} E_n t} = \text{Tr} e^{-\frac{i}{\hbar} \hat{H} t} = \sum_n \langle n | e^{-\frac{i}{\hbar} \hat{H} t} | n \rangle$$

ANALYTIC CONTINUATION TO IMAGINARY TIME

$$\beta \equiv \frac{it}{\hbar}$$

$$\int dx K(x, t; x, 0) = \sum_n e^{-\beta E_n} = \text{Tr} e^{-\beta \hat{H}}$$

"PARTITION FUNCTION"

FOR  $\beta \in \mathbb{R}$  AND POSITIVE THIS CORRESPONDS TO STATISTICAL MECHANICS PROBLEM ( $\beta$  IS  $\frac{1}{k_B T}$  WITH  $T$ : TEMPERATURE)

IN LIMIT  $\beta \rightarrow \infty$  (ZERO TEMPERATURE LIMIT) ONLY GROUND STATE CONTRIBUTES TO SUM

$$\int dx K(x, -it/\hbar; x, 0) \xrightarrow{\beta \rightarrow \infty} e^{-\beta E_0}$$

GROUND STATE ENERGY (FEYNMAN - KAC FORMULA)

$$E_0 = \lim_{\beta \rightarrow \infty} \left(-\frac{1}{\beta}\right) \ln \int dx K(x, -it/\hbar; x, 0) = \lim_{\beta \rightarrow \infty} \left(-\frac{1}{\beta}\right) \ln \text{Tr} e^{-\beta \hat{H}}$$

→ EXAMPLE : HARMONIC OSCILLATOR.

$$\hookrightarrow K(x, t; x', 0)$$

$$= \left( \frac{m\omega}{2\pi i \hbar \sin \omega t} \right)^{1/2} e^{\frac{i}{\hbar} S_{cl}}$$

$$\text{WITH } S_{cl} = \frac{m\omega}{2 \sin \omega t} \left\{ (x^2 + x'^2) \cos \omega t - 2xx' \right\}$$

$$\hookrightarrow \int dx K(x, t; x, 0)$$

$$= \left( \frac{m\omega}{2\pi i \hbar \sin \omega t} \right)^{1/2} \int dx e^{-ax^2}$$

$$\downarrow \text{ WITH } a = -\frac{i}{\hbar} \frac{m\omega}{\sin \omega t} (\cos \omega t - 1)$$

$$= \frac{i}{\hbar} (m\omega) \frac{\sin(\omega t/2)}{\cos(\omega t/2)}$$

$$= \left( \frac{m\omega}{2\pi i \hbar \sin \omega t} \right)^{1/2} \left( \frac{\pi \hbar \cos(\omega t/2)}{i m \omega \sin(\omega t/2)} \right)^{1/2}$$

$$2 \sin \frac{\omega t}{2} \cos \frac{\omega t}{2}$$

$$= \frac{1}{2i} \frac{1}{\sin(\omega t/2)}$$

$$E_0 = \lim_{\beta \rightarrow \infty} \left( -\frac{1}{\beta} \right) \ln \int dx \cdot K(x, -i\hbar\beta; x, 0)$$

$$= \lim_{\beta \rightarrow \infty} \left( -\frac{1}{\beta} \right) \ln \left[ \frac{1}{2i} \frac{1}{\sin(-i\frac{\hbar\omega}{2}\beta)} \right]$$

$$\downarrow \quad \frac{1}{\sin x} = \frac{2i e^{-ix}}{1 - e^{-2xi}}$$

$$= \lim_{\beta \rightarrow \infty} \left( -\frac{1}{\beta} \right) \ln \left[ \frac{e^{-\beta \frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}} \right]$$

$$= \lim_{\beta \rightarrow \infty} \left( -\frac{1}{\beta} \right) \left\{ -\beta \frac{\hbar\omega}{2} + O(e^{-\beta\hbar\omega}) \right\}$$

$$\underline{\underline{E_0 = \frac{\hbar\omega}{2}}}$$

↳ IN GENERAL : PARTITION FUNCTION

$$\int dx \cdot K(x, -i\hbar\beta; x, 0) = \sum_n e^{-\beta E_n}$$

$$\frac{e^{-\beta \frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}} = e^{-\beta \frac{\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-\beta n \hbar\omega}$$

⇓

$$\underline{\underline{E_m = \hbar\omega \left( m + \frac{1}{2} \right)}}$$



→ GROUND STATE WAVE FUNCTION

$$K(x, -i\hbar\beta; x, 0) = \sum_m e^{-\beta E_m} |\psi_m(x)|^2$$

$$\xrightarrow{\beta \rightarrow \infty} e^{-\beta E_0} |\psi_0(x)|^2$$

H.O.:  $K(x, -i\hbar\beta; x, 0)$

$$= \left( \frac{m\omega}{\pi\hbar 2i \sin(-i\hbar\omega\beta)} \right)^{1/2} \exp \left\{ \frac{-i}{\hbar} m\omega x^2 \tan\left(-i\frac{\hbar\omega\beta}{2}\right) \right\}$$

$$\frac{1}{2i \sin(-i\hbar\omega\beta)} = \frac{e^{-\hbar\omega\beta}}{1 - e^{-2\hbar\omega\beta}} \downarrow$$

$$i \tan\left(-i\frac{\hbar\omega\beta}{2}\right) = \frac{1 - e^{-\hbar\omega\beta}}{1 + e^{-\hbar\omega\beta}}$$

$$= \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{e^{-\beta E_0}}{\left(1 - e^{-\beta 2\hbar\omega}\right)^{1/2}} \exp \left\{ -\frac{m\omega}{\hbar} x^2 \left[ 1 + O(e^{-\beta\hbar\omega}) \right] \right\}$$

$$\xrightarrow{\beta \rightarrow \infty} e^{-\beta E_0} \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \exp \left\{ -\frac{m\omega}{\hbar} x^2 \right\}$$

$$\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left\{ -\frac{m\omega}{2\hbar} x^2 \right\}$$

GROUND STATE W.F. OF H.O.

### 3) EXAMPLE IN 3D: AHARONOV-BOHM EFFECT

- CONSIDER  $e^-$  MOVING IN MAGNETIC FIELD

→ HAMILTONIAN  $H = \frac{1}{2m} (\bar{p} - \frac{e}{c} \bar{A})^2$   
 (MINIMAL SUBSTITUTION)

→ LAGRANGE FUNCTION  $L = \bar{p} \cdot \dot{\bar{q}} - H$   
 $\bar{q}$ : COORDINATE,  $\dot{\bar{q}}$  VELOCITY  
 $\bar{p}$ : CONJUGATE MOMENTUM

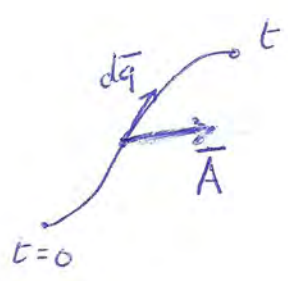
$\bar{p} = \frac{\partial L}{\partial \dot{\bar{q}}}$   
 $\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} = \frac{1}{m} (\bar{p} - \frac{e}{c} \bar{A})$

→ CANONICAL MOMENTUM  $\bar{p} = \underbrace{m \dot{\bar{q}} + \frac{e}{c} \bar{A}}_{\text{KINETIC MOMENTUM}}$

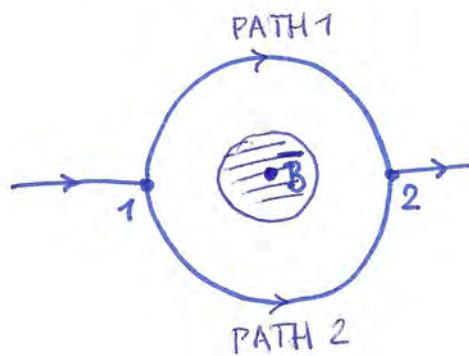
→  $L = (m \dot{\bar{q}} + \frac{e}{c} \bar{A}) \cdot \dot{\bar{q}} - \frac{m}{2} \dot{\bar{q}}^2$

$L = \frac{1}{2} m \dot{\bar{q}}^2 + \frac{e}{c} \dot{\bar{q}} \cdot \bar{A}$

→ ACTION  $S = \int_0^t dt' L(\bar{q}, \dot{\bar{q}})$   
 $= \frac{m}{2} \int_0^t dt' \frac{d\bar{q}}{dt'} \cdot \frac{d\bar{q}}{dt'} + \frac{e}{c} \int_0^t dt' \frac{d\bar{q}}{dt'} \cdot \bar{A}$   
 $= \frac{m}{2} \int_{\text{PATH}} d\bar{q} \cdot \frac{d\bar{q}}{dt'} + \frac{e}{c} \int_{\text{PATH}} d\bar{q} \cdot \bar{A}$



→ CONSIDER CONDUCTING RING  
CURRENT ENTERS AT POINT 1, IS EXTRACTED AT POINT 2



CONDUCTING WIRE AROUND SOLENOID →  $\vec{B}$ -FIELD

→ PROBABILITY AMPLITUDE FOR ELECTRON  
TO BE OBSERVED AT POINT 2

$$K(2, 1) \sim e^{\frac{i}{\hbar} S(\text{PATH 1})} + e^{\frac{i}{\hbar} S(\text{PATH 2})}$$

$$= e^{\frac{i}{\hbar} S(\text{PATH 1})} \left( 1 + e^{\frac{i}{\hbar} \Delta S} \right)$$

$$\Delta S = S(\text{PATH 2}) - S(\text{PATH 1})$$

$$= \frac{e}{c} \left\{ \int_{\text{PATH 2}} d\vec{q} \cdot \vec{A} - \int_{\text{PATH 1}} d\vec{q} \cdot \vec{A} \right\}$$

$$= \frac{e}{c} \oint d\vec{q} \cdot \vec{A}$$

NOTE : KINETIC ENERGY TERMS ARE EQUAL  
DUE TO SYMMETRY → AND CANCEL IN DIFFERENCE

## STOKES THEOREM

$$\Delta S = \frac{e}{c} \int d\vec{S} \cdot (\vec{\nabla} \times \vec{A})$$

$d\vec{S}$  : NORMAL TO SURFACE OF CURRENT LOOP

$$= \frac{e}{c} \int d\vec{S} \cdot \vec{B}$$

$$\Delta S = \frac{e}{c} \Phi$$

↳ MAGNETIC FLUX

∴ PROB. AMPLITUDE  $\sim 1 + e^{\frac{ie}{\hbar c} \Phi}$

$$\text{PROBABILITY} = |K(2,1)|^2$$

$e^{\frac{ie}{\hbar c} \Phi}$  TERM WILL GIVE INTERFERENCE PATTERN WHICH WILL CHANGE BY VARYING  $\Phi$  (i.e.  $\vec{B}$ )

CONSTRUCTIVE INTERFERENCE

$$\frac{e\Phi}{\hbar c} = 2\pi m \Rightarrow \Phi = \left(\frac{\hbar c}{e}\right) m = m \phi_0$$

$$m \in \mathbb{Z}$$

$$\phi_0 = \frac{\hbar c}{e}$$

QUANTUM OF FLUX

MAXIMUM CURRENT FOR  $\Phi$  EQUAL TO AN INTEGER TIMES  $\phi_0$  : AHARONOV - BOHM EFFECT

↳ WAS FIRST OBSERVED EXPERIMENTALLY BY CHAMBERS (1960)

# 4) BROWNIAN MOTION AND WIENER PATH INTEGRAL

## • RANDOM WALK IN 1 DIMENSION

↳ CONSIDER DISCRETE RANDOM WALK

DISCRETE TIME STEPS  $\epsilon$

DISCRETE SPATIAL STEPS  $l$  EITHER TO LEFT OR TO RIGHT

SUPPOSE : INITIALLY  $t=0$  , POSITION  $x=0$



↳ PROBABILITY FOR LEFT STEP AND RIGHT STEP  $\Rightarrow$  EACH  $\frac{1}{2}$   
 SUCCESSIVE STEPS ARE STATISTICALLY INDEPENDENT  
 (STOCHASTIC PROCESS)

PROBABILITY FOR TRANSITION  
 FROM  $x = j l$  TO  $x = i l$  DURING TIME  $\epsilon$   
 $(i, j \in \mathbb{Z})$

$$\| P(i l - j l, \epsilon) = \begin{cases} \frac{1}{2} , & |i - j| = 1 \\ 0 , & \text{OTHERWISE} \end{cases}$$

- HOMOGENEOUS IN SPACE : ONLY DEPENDS ON  $i - j$
- ISOTROPIC IN SPACE : SYMMETRIC UNDER  $(i, j) \rightarrow (-i, -j)$

↳ DISCRETE RANDOM WALK: EXAMPLE OF MARKOV CHAIN

MARKOV CHAIN: CHARACTERIZED BY  $(P(t_m), P(0))$

$P_{ij}(t_m)$ : TRANSITION PROBABILITY FROM  $j \rightarrow i$  AT TIME  $t_m$

$P_i(0)$ : INITIAL PROBABILITY DISTRIBUTION

$$\| P_i(t_m) = \sum_j P_{ij}(t_m) P_j(0)$$

NOTE  $0 \leq P_i(0) \leq 1$   $\sum_i P_i(0) = 1$

$$0 \leq P_{ij} \leq 1 \quad \sum_i P_{ij} = 1$$

|| TRANSITION PROB. TO STATE  $i$  AT TIME  $t_m$   
 DEPENDS ONLY ON STATE  $j$  AT TIME  $t_{m-1}$   
 AND NOT ON STATES AT EARLIER TIMES  $t_{m-2}, t_{m-3}, \dots$

FOR DISCRETE RANDOM WALK.

$$P_{ij}(\varepsilon) = P(i\ell - j\ell, \varepsilon)$$

MARKOV CHAIN: SUCCESSIVE STEPS STATISTICALLY INDEPENDENT

CHOOSE  $P_j(0) = \delta_{j0}$ , i.e.  $x(0) = 0$

↳ P IN MATRIX NOTATION

MATRIX ELEMENT  $ij$

$$P(\varepsilon) = \frac{1}{2} (R(\varepsilon) + L(\varepsilon))$$

$$R(\varepsilon) : \text{STEP TO RIGHT} \quad (R(\varepsilon))_{ij} = \delta_{i, j+1}$$

$$L(\varepsilon) : \text{STEP TO LEFT} \quad (L(\varepsilon))_{ij} = \delta_{i, j-1}$$

e.g.

$$\begin{aligned} P_i(\varepsilon) &= \sum_j P_{ij}(\varepsilon) P_j(0) \\ &= \frac{1}{2} \sum_j (R_{ij}(\varepsilon) + L_{ij}(\varepsilon)) \delta_{j0} \\ &= \frac{1}{2} (R_{i0}(\varepsilon) + L_{i0}(\varepsilon)) \\ &= \frac{1}{2} (\delta_{i1} + \delta_{i-1}) \end{aligned}$$

↳ AFTER  $m$  TIME STEPS

$$P_i(m\varepsilon) = \sum_j (P^m(\varepsilon))_{ij} \underbrace{P_j(0)}_{\delta_{j0}}$$

NOTE  $P = \frac{1}{2} (R + L)$

$$P^m = \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} R^k L^{m-k}$$

(BINOMIAL FORMULA)

$$AS \quad RL = LR = 1$$

$$\downarrow$$

$$P^n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} R^{2k-n}$$

$$(P^n)_{ij} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \underbrace{(R^{2k-n})}_{\delta_{i, j+2k-n}}_{ij}$$

$$(P^n)_{ij} = \begin{cases} \frac{1}{2^n} \binom{n}{\frac{1}{2}(i-j+n)} & , \text{ if } |i-j| \leq n \\ & \text{ AND } i-j+n \text{ EVEN} \\ 0 & , \text{ OTHERWISE} \end{cases}$$

$$\Downarrow$$

$$P_i(m\varepsilon) = \sum_j (P^m)_{ij} P_j(0)$$

$$= (P^m)_{i0}$$

$$= \begin{cases} \frac{1}{2^m} \binom{m}{\frac{1}{2}(i-m)} & , \text{ if } |i| \leq m \\ & \text{ AND } i-m \text{ EVEN} \\ 0 & , \text{ OTHERWISE} \end{cases}$$

### NOTE

$$P(i\ell - j\ell, m\varepsilon) = (P^{(m\varepsilon)})_{ij}$$

### PROPERTIES:

- 1) HOMOGENEOUS IN SPACE : DEPENDS ONLY ON  $i - \ell$
- 2) " " TIME : DOES NOT DEPEND ON INITIAL TIME, ONLY ON TIME DIFFERENCE  $m$
- 3) ISOTROPIC IN SPACE  $P(-i\ell + j\ell, m\varepsilon) = P(i\ell - j\ell, m\varepsilon)$



↳ RECURSION FORMULA FOR BINOMIAL COEFF

$$\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$$

PROOF:

$$\text{LEFT} = \frac{(m+1)!}{(m+1-k)! k!}$$

$$\text{RIGHT} = \frac{m!}{(m-k)! k!} + \frac{m!}{(m-k+1)! (k-1)!}$$

$$= \frac{m!}{(m+1-k)! k!} (m+1-k+k) \stackrel{!}{=} \frac{(m+1)!}{(m+1-k)! k!}$$

AS 
$$P(i\ell - j\ell, n\varepsilon) = \frac{1}{2^n} \binom{m}{k}$$

FOR  $k = \frac{1}{2} (i-j+m)$

(IF)  $|i-j| \leq m$   
 $i-j+m$  EVEN

$$P(i\ell - j\ell, (m+1)\varepsilon)$$

$$= \frac{1}{2} \frac{1}{2^m} \binom{m}{k} + \frac{1}{2} \frac{1}{2^m} \binom{m}{k-1}$$

$$k = \frac{1}{2} (i-j+m+1)$$

$$= \frac{1}{2} (i-j+1+m)$$

$$k-1 = \frac{1}{2} (i-j+m-1)$$

$$= \frac{1}{2} (i-j-1+m)$$

$$= \frac{1}{2} P((i-j+1)\ell, m\varepsilon) + \frac{1}{2} P((i-j-1)\ell, m\varepsilon)$$

OR FOR  $x = (i-j)l$   
 $t = n\varepsilon$

$$P(x, t+\varepsilon) = \frac{1}{2} P(x+l, t) + \frac{1}{2} P(x-l, t)$$

$$\frac{1}{\varepsilon} \left( P(x, t+\varepsilon) - P(x, t) \right)$$

$$= \frac{l^2}{2\varepsilon} \cdot \frac{1}{l^2} \left( P(x+l, t) - 2P(x, t) + P(x-l, t) \right)$$



MACROSCOPIC DESCRIPTION OF RANDOM WALK

LIMIT  $\varepsilon \rightarrow 0$

$l \rightarrow 0$

$$\frac{l^2}{2\varepsilon} = D \quad \text{FINITE}$$

$$\frac{\partial}{\partial t} P(x, t) = D \frac{\partial^2}{\partial x^2} P(x, t)$$

DIFFUSION EQUATION

$D$  : DIFFUSION CONSTANT

↳ SOLUTION OF DIFFUSION EQ.

INITIAL CONDITION

$$t=0 \Rightarrow x=0 : P(x,0) = \delta(x)$$

$$\int dx P(x,0) = 1$$

$$P(x,t) = \int dk e^{ikx} \tilde{P}(k,t)$$

$$\tilde{P}(k,t) = \overset{\uparrow}{\int} \frac{dx}{2\pi} e^{-ikx} P(x,t)$$

$$\tilde{P}(k,0) = \frac{1}{2\pi}$$

$$\circ \circ \quad \frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t)$$

↓

$$\frac{\partial}{\partial t} \tilde{P}(k,t) = -Dk^2 \tilde{P}(k,t)$$

$$\frac{\partial}{\partial t} \ln \tilde{P}(k,t) = -Dk^2$$

$$\Downarrow \int_0^t$$

$$\ln \frac{\tilde{P}(k,t)}{\tilde{P}(k,0)} = -Dk^2 t$$

$$\boxed{\tilde{P}(k,t) = \frac{1}{2\pi} \exp(-Dk^2 t)}$$

$$\begin{aligned}
 \rightsquigarrow P(x,t) &= \int_{-\infty}^{+\infty} dk e^{ikx} \frac{1}{2\pi} e^{-Dk^2 t} \\
 &= \frac{1}{2\pi} \int dk e^{-Dt \left(k - \frac{ix}{2Dt}\right)^2 - \frac{x^2}{4Dt}} \\
 &= \frac{1}{2\pi} \sqrt{\frac{\pi}{Dt}} e^{-\frac{x^2}{4Dt}}
 \end{aligned}$$

$$P(x,t) = \left( \frac{1}{4\pi Dt} \right)^{1/2} e^{-\frac{x^2}{4Dt}}$$

SATISFIES  $\int dx P(x,t) = 1$

MEAN SQUARE DEVIATION

$$\begin{aligned}
 \langle x^2 \rangle_t &= \int dx x^2 P(x,t), \quad \text{NOTE } \langle x \rangle = 0 \\
 &= 2Dt
 \end{aligned}$$

$$\Downarrow \\
 \underline{\underline{\sqrt{\langle x^2 \rangle_t} = \sqrt{2Dt}}}$$

$\hookrightarrow$  FOR ARBITRARY INITIAL CONDITION  $x(t_0) = x_0$

$$\begin{aligned}
 &P(x,t; x_0, t_0) \\
 &= \left( \frac{1}{4\pi D(t-t_0)} \right)^{1/2} e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}
 \end{aligned}$$

GAUSSIAN

PROBABILITY FOR RANDOM WALKER  
WHICH INITIALLY STARTS AT  $x_0, t_0$   
TO BE FOUND AT A LATER TIME  $t$  AT  $x$

## • IMAGINARY TIME

↳ DIFFUSION EQ.

$$D \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial t}$$

⇓ ANALYTICALLY CONTINUE TO  
IMAGINARY TIME i.e.  $t \Rightarrow it$

$$D \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial(it)}$$

$$- D \frac{\partial^2 P}{\partial x^2} = i \frac{\partial P}{\partial t}$$

FOR  $D = \frac{\hbar}{2m}$  THIS IS SCHRÖDINGER EQ. !

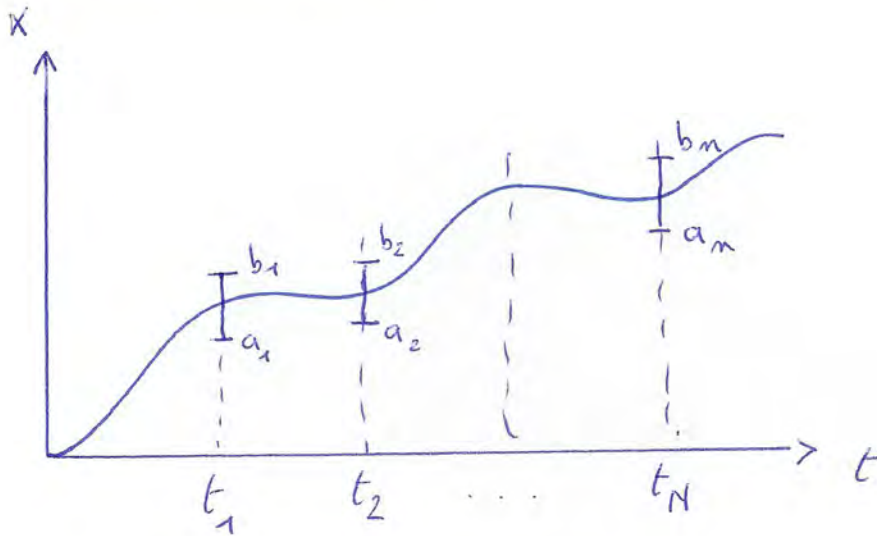
$$\begin{aligned} \text{↳ } P(x, it; x_0, it) \Big|_{D = \frac{\hbar}{2m}} \\ = \left( \frac{m}{2\pi\hbar i(t-t_0)} \right)^{1/2} \exp \left\{ + \frac{im}{2\hbar} \frac{(x-x_0)^2}{(t-t_0)} \right\} \end{aligned}$$

NOTE 1) THIS IS PRECISELY KERNEL  $K(x, t; x_0, t)$   
FOR A FREE PARTICLE IN QUANTUM MECHANICS !

2) DIFFUSION EQ : GAUSSIAN  $P$  (PROBABILITY)  
SCHRÖDINGER EQ : OSCILLATORY KERNEL (PROB. AMPLITUDE!)  
⇓  
→ ADMITS WAVE SOLUTIONS.  
→ INTERFERENCE EFFECTS

3) QM FREE PARTICLE SOLUTION MAY BE SEEN  
AS ANALYTIC CONTINUATION TO IMAG. TIME OF INDETERMINISTIC  
MOTION OF BROWNIAN PARTICLE

• WIENER PATH INTEGRAL (1921)



↳ CONSIDER BROWNIAN PARTICLE

PROBABILITY TO FIND BROWNIAN PARTICLE

WHICH STARTED AT  $t=0$  IN  $x=0$

AT TIME  $t_1$  IN INTERVAL  $[a_1, b_1]$

AT TIME  $t_2$  IN INTERVAL  $[a_2, b_2]$

⋮  
AT TIME  $t_N$  IN INTERVAL  $[a_N, b_N]$

PRODUCT OF PROBABILITIES DUE TO STOCHASTIC PROCESS

$$\text{PROB} \left\{ x(t_1) \in [a_1, b_1], x(t_2) \in [a_2, b_2], \dots, x(t_N) \in [a_N, b_N] \right\}$$

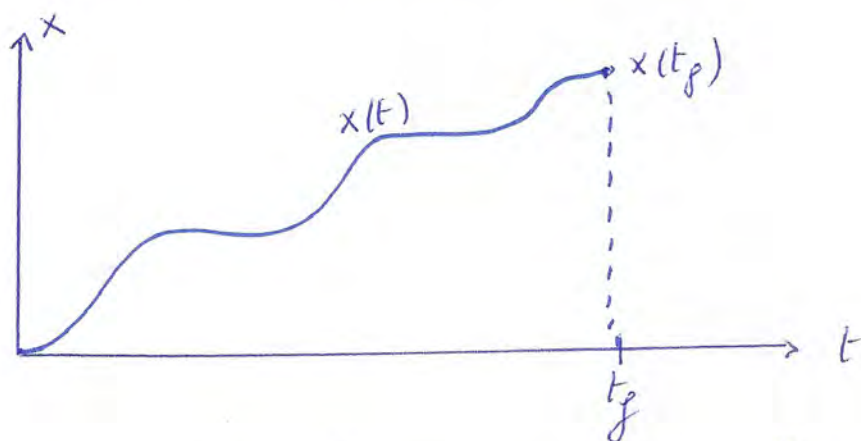
$$= \int_{a_1}^{b_1} dx_1 P(x_1, t_1; 0, 0) \int_{a_2}^{b_2} dx_2 P(x_2, t_2; x_1, t_1)$$

$$\dots \int_{a_N}^{b_N} dx_N P(x_N, t_N; x_{N-1}, t_{N-1})$$

WITH  $P(x_{i+1}, t_{i+1}; x_i, t_i) = \left( \frac{1}{4\pi D(t_{i+1} - t_i)} \right)^{1/2} e^{-\frac{(x_{i+1} - x_i)^2}{4D(t_{i+1} - t_i)}}$

↳ CONSIDER  $t_i - t_{i-1} = \Delta t_i \rightarrow 0$  (i.e.  $N \rightarrow \infty$ )  
 $x_i - x_{i-1} = dx_i \rightarrow 0$

PROBABILITY THAT BROWNIAN PARTICLE  
 MOVES ALONG A TRAJECTORY  $x(t)$   
 FROM  $x=0$  AT  $t=0$  TO  $x(t_f)$ .

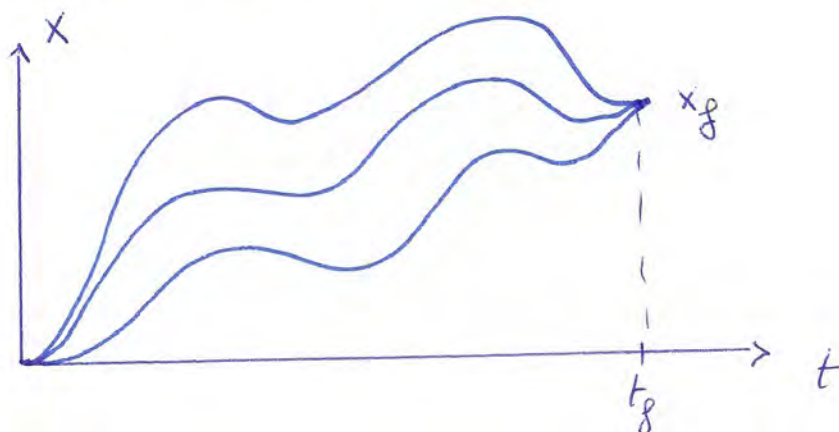


$$P = \lim_{\substack{\Delta t_i \rightarrow 0 \\ N \rightarrow \infty}} \exp \left\{ - \sum_{i=1}^N \frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})} \right\} \prod_{i=1}^N \left( \frac{1}{4\pi D(t_i - t_{i-1})} \right)^{1/2} dx_i$$

$$= \lim_{\substack{\Delta t_i \rightarrow 0 \\ N \rightarrow \infty}} \exp \left\{ - \sum_{i=1}^N \left( \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right)^2 \frac{\Delta t_i}{4D} \right\} \prod_{i=1}^N \left( \frac{1}{4\pi D \Delta t_i} \right)^{1/2} dx_i$$

$$= \exp \left\{ - \frac{1}{4D} \int_0^{t_f} \dot{x}^2(t) dt \right\} \prod_{t=0}^{t_f} \frac{dx(t)}{\sqrt{4\pi D dt}}$$

↳ PATH INTEGRAL (WIENER)



$$P(x_f, t_f; 0, 0)$$

CAN BE OBTAINED IN 2 WAYS

1) AS BEFORE

$$P(x_f, t_f; 0, 0) = \left( \frac{1}{4\pi D t_f} \right)^{1/2} e^{-\frac{x_f^2}{4D t_f}}$$

2) AS SUM OF PROBABILITIES OF ALL PATHS  
CONNECTING  $x(0) = 0$  AND  $x(t_f) = x_f$

$$P(x_f, t_f; 0, 0) = \int_{C\{0,0; x_f, t_f\}} d_W x(t)$$

WITH  $d_W x(t)$  : WIENER MEASURE

$$d_W x(t) \equiv \exp \left\{ -\frac{1}{4D} \int_0^{t_f} dt \dot{x}^2(t) \right\} \prod_{t=0}^{t_f} \frac{dx(t)}{\sqrt{4\pi D dt}}$$

WELL DEFINED INTEGRAL



↳ FEYNMAN PATH INTEGRAL (FREE PARTICLE)

$$K(x, t_f; 0, 0) \sim \exp \left\{ \frac{i}{\hbar} \int_0^{t_f} dt \frac{1}{2} m \dot{x}^2 \right\}$$

↓  
ANALYTICALLY CONTINUE TO  
IMAG TIME  $\tau = it$

$$K(x, it_f; 0, 0) \sim \exp \left\{ \frac{1}{\hbar} \int_0^{it_f} d(it) \frac{1}{2} m \dot{x}^2 \right\}$$

↓  $\dot{x}^2 = - \left( \frac{dx}{d\tau} \right)^2$

$$K(x, \tau; 0, 0) \sim \exp \left\{ - \frac{m}{2\hbar} \int_0^{\tau_f} d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}$$

$$= \exp \left\{ - \frac{1}{4D} \int_0^{\tau_f} d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}$$

WITH  $D = \frac{\hbar}{2m}$