Interplay Between Low-Energy and High-Energy Behavior in Dispersive Sum Rules for Compton and Light-by-Light Scattering

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Chapter 1

Introduction

Causality is the fundamental principle in deriving sum rules: a signal can be influenced only by signals lying in the past light cone. As an immediate consequence of causality, a scattering amplitude must be an analytic function in the complex energy plane. Based on this general property of the scattering amplitude, relations between physical observables can be inferred, in particular sum rules. These relations are important tools in understanding the general constraints of a theory, especially when a perturbative approach is unsatisfactory, e.g. the strong interaction or the theory of bound states.

In particle physics, causality together with the unitary property of the scattering matrix $\mathcal{S}$ imply that the scattering amplitude $\mathscr{M}(s,t,u)$ of a process, where s, t, and u are the Mandelstam variables, is analytic in the complex plane, excepting a set of singularities and branch-cuts, which have physical interpretation [1, p. 276]. Based on these, rather loose, constrains on $\mathscr{M}(s,t,u)$, Gell-Mann, Goldberger, and Thirring published a dispersion relation for the forward Compton scattering [2]. After that, supported by the low energy theorem [3; 4], a series of famous sum rules were developed: the Baldin sum rule (Baldin-SR) [5], the Gerasimov-Drell-Hearn sum rule [6; 7], the Burkhardt-Cottingham sum rule [8], and the Schwinger sum rule [9]. In recent years, [10] proposed three sum rules for the light-by-light scattering, and in [11], the Schwinger sum rule is employed to estimate the hadronic contributions to anomalous magnetic moment of the muon.

Sucher [12], and Bernabéu and Tarrach [13] introduced an unsubtracted sum rule for the longitudinal polarization of a photon interacting with a nucleon, hereinafter referred to as longitudinal sum rule (LSR). This sum rule should link the electric dipole polarizability α_{E1} of a particle to the longitudinal photo-absorption cross-section [14]. LSR in combination with Baldin-SR would allow the independent assessment of the electric α_{E1} and magnetic β_{M1} dipole polarizabilities of particles. Llanta and Tarrach [15] proved that LSR holds up to a constant in quantum-electrodynamics (QED) for spin-½ particles. In consequence, the LSR should be rewritten as a subtracted dispersion relation. The need of a subtraction point made LSR inappropriate for apprising α_{E1} and β_{M1} because the subtraction required is a function of the energy-momentum transfer Q^2 of the photon.

1.1 Outline of the Thesis

The current paper revisits LSR in QED to leading order in perturbation theory for the cases of the interaction between a photon field and a vector or a scalar neutral field, respectively, involving a fermion field or a charged scalar field, see Table 1.1. This is a framework slightly different from the one used in [15], see Appendix A for the Feynman diagrams used by Llanta and Tarrach. As a novelty, it is shown that there exist interaction Lagrangians for which LSR holds, at least in leading order. For the other interaction Lagrangians studied, LSR holds up to a constant, denoted by Δ_{L}, a result compatible with [15]. An interpretation of Δ_{L} is given in the light of the Sugawara-Kanazawa theorem [16], and an ultra-violet (UV) completion of the interaction Lagrangians is proposed such that LSR is fulfilled exactly, i.e. $\Delta_{\mathrm{L}} = 0$. Based on these insights, an UV completion, in the from of a Nambu-Jona-Lasinio interaction, is suggested for the theory analysed in [15].

Table 1.1: **List of Lagrangians analyzed in the current paper**. m is the mass of the fermion field ψ, or charged scalar field π, depending on the context. M is the mass of the scalar field ϕ, or of the massive vector field B_μ, depending on the context. A_μ is the photon field. e is the electric coupling. g is the coupling of ϕ or B_μ to ψ or π, also context dependent. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

#	Lagrangian
1.	$\mathscr{L}_1 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
2.	$\mathscr{L}_2 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + \mathrm{i}g\bar{\psi}\gamma_5\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
3.	$\mathscr{L}_3 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
4.	$\mathscr{L}_4 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\psi B_\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
5.	$\mathscr{L}_5 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
6.	$\mathscr{L}_6 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\sigma^{\mu\nu}\psi G_{\mu\nu} - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
7.	$\mathscr{L}_7 = \left[(\partial_\mu + \mathrm{i}eA_\mu)\pi\right]^\dagger[(\partial^\mu + \mathrm{i}eA^\mu)\pi] + g\pi^\dagger\pi\phi - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
8.	$\mathscr{L}_8 = \left[(\partial_\mu + \mathrm{i}eA_\mu + \mathrm{i}gB_\mu)\pi\right]^\dagger[(\partial^\mu + \mathrm{i}eA^\mu + \mathrm{i}gB^\mu)\pi] - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$

Notable is also that, a light-by-light interaction involving fermions or charged scalars could allow a super-convergent sum rule or the estimation of Δ_L by using the cross-section when both photons are longitudinal polarized and the cross-section when one photon is transversal polarized and the other is longitudinal polarized.

In Chapter 2 the optical theorem and the mathematical foundations of dispersion relations are introduced. Chapter 3 analysis the unsubtracted sum rule for longitudinal polarization of the photon (LSR) in leading order QED for a set of interaction Lagrangians. To set up the stage, a short discussion to cross-sections, forward double virtual Compton scattering (FVVCS), dispersion relations, and sum rules is given. A reformulation of the LSR is proposed in Chapter 3.3. The calculations preformed are exemplified in Chapter 3.4, and the main results are discussed in Chapter 3.5. Chapter 4 examines the photo-disintegration process of a vector particle. The results are reviewed in Chapter 4.2, manly by considering the LSR implications in light-by-light scattering case. Chapter 5 concludes this work and advocates for future efforts regarding LSR.

1.2 Conventions and Notations

This section aims to list the conventions and notations that are used throughout the paper. They follow the current consensus in the scientific community, at least for the foundations of quantum field theory (QFT).

In this paper the natural units are used, where the speed of light and the Plank constant have the numeric value 1, i.e. $c = \hbar = 1$. Einstein summation convention states that terms indexed with the same indices connote a summation, e.g.:

$$u^{imj}v^{kml} = \sum_m u^{imj}v^{kml}$$

The contravariant metric tensor of the underlying Minkowski space is denoted by $g^{\mu\nu}$ and it is defined by

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The covariant metric tensor $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$, thus with the same matrix representation as $g^{\mu\nu}$ from above. Given the metric tensor $g^{\mu\nu}$, the scalar product of two four-vectors x^μ and y^ν has the form

$$x \cdot y = x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu = x^0 y^0 - \mathbf{x} \cdot \mathbf{y}.$$

For brevity, $x^2 = x \cdot x$. A four-vector x is said to be: (i) time-like if $x^2 > 0$; (ii) space-like if $x^2 < 0$; (iii) light-like if $x^2 = 0$.

Because $g^{\mu\nu}$ and $g_{\mu\nu}$ have the same symmetric matrix representation, the following tensor notations are equivalent:

$$A^\mu_\nu = A^\mu{}_\nu = A_\nu{}^\mu = \cdots \tag{1.1}$$
$$A^{\mu\sigma}_\nu = A^\mu{}_\nu{}^\sigma = A_\nu{}^{\mu\sigma} = \cdots . \tag{1.2}$$

For matrix elements, gamma matrices and Feynman rules the conventions used in [17, p. 801] are adopted.

As a rule of thumb, the following notations are widely used in this work:

1. p is the four-momentum of a massive particle with mass M,
2. q is the four-momentum of a photon with energy-momentum transfer $Q^2 = -q^2$,
3. k is an unbounded four-momentum in a loop diagram and/or integral,
4. A_μ is the photon field,
5. ψ is a fermion field,
6. π is a charged scalar field,
7. ϕ is a scalar field,
8. B_μ is a massive vector field
9. m is the mass of a fermion ψ or the mass of a charged scalar π, depending on the context,
10. M is the mass of a scalar ϕ or the mass of a vector B_μ, depending on the context,
11. e is the charge of a fermion ψ or the charge of a charged scalar π, depending on the context,
12. g is the coupling of a scalar ϕ or coupling of a vector B_μ to a fermion ψ or a charged scalar π, depending on the context,
13. s is the energy of the system in the center-of-momentum reference system (COM),
14. $\nu = \frac{p.q}{M}$ is the fractional energy of the photon.

Chapter 2

Mathematical Methods

Dispersive sum rules emerge from combining the optical theorem with the analyticity property of the scattering amplitude. The optical theorem is an immediate result of the unitarity of the scattering matrix $\mathcal{S}$. The analyticity of the scattering amplitude has its roots in the causality principle.

2.1 The Optical Theorem

Conservation of probability is postulated in QFT. It claims that any state $|i\rangle$ of a system can be expressed at a later time as a combination of the complete set of states that characterizes the system [1, p. 181]. This claim applies also to asymptotic times $t \to \infty$, therefore the scattering matrix S is unitary

$$\mathcal{S}^\dagger \mathcal{S} = 1. \tag{2.1}$$

In terms of the transition matrix $\mathcal{T} = \mathcal{S} - \mathbb{1}$, the above expression becomes

$$\mathrm{i}(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger \mathcal{T}. \tag{2.2}$$

Consider $|i\rangle$ an initial state, with total four-momentum p_i, and $|f\rangle$ a final state, with total four-momentum p_f. The invariant matrix element $\mathcal{M}$ of the transition amplitude between $|i\rangle$ and $|f\rangle$ is given by $\langle f|\mathcal{T}|i\rangle = (2\pi)^4\delta^4(p_\mathrm{i} - p_\mathrm{f})\mathcal{M}(i \to f)$. This equality motivates the application of $|i\rangle$ and $|f\rangle$ to both sides of Eq. 2.2, and the introduction in the RHS of Eq. 2.2 of a complete set of Hilbert-states of the system, thus

$$\begin{aligned}
\langle f|\mathrm{i}(\mathcal{T}^\dagger - \mathcal{T})|i\rangle &= \mathrm{i}\langle i|\mathcal{T}|f\rangle^* - \mathrm{i}\langle f|\mathcal{T}|i\rangle && (2.3)\\
&= \mathrm{i}(2\pi)^4\delta^4(p_\mathrm{i} - p_\mathrm{f})\big(\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f)\big), && (2.4)\\
\langle f|\mathcal{T}^\dagger\mathcal{T})|i\rangle &= \sum_\mathrm{X}\int \mathrm{d}\Pi_\mathrm{X}\langle f|\mathcal{T}^\dagger|\mathrm{X}\rangle\langle \mathrm{X}|\mathcal{T}|i\rangle && (2.5)\\
&= \sum_\mathrm{X}\mathrm{d}\Pi_\mathrm{X}(2\pi)^8\delta^4(p_\mathrm{i} - p_\mathrm{X})\delta^4(p_\mathrm{f} - p_\mathrm{X})\int \mathcal{M}(i \to \mathrm{X})\mathcal{M}^*(f \to \mathrm{X}). && (2.6)
\end{aligned}$$

By combining the above two relations with Eq. 2.2, the generalized optical theorem is obtained

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = \mathrm{i}\sum_\mathrm{X}\int \mathrm{d}\Pi_\mathrm{X}(2\pi)^4\delta^4(p_\mathrm{i} - p_\mathrm{X})\mathcal{M}(i \to \mathrm{X})\mathcal{M}^*(f \to \mathrm{X}). \tag{2.7}$$

Note that Eq. 2.7 is valid at any order in perturbation theory [18, p. 454].

A special case of the generalized optical theorem is when the initial state $|i\rangle$ and the final state $|f\rangle$ can be identified with same two-particle state, i.e. $|i\rangle = |f\rangle = |p_1, p_2\rangle$, see Fig. 2.1. For this case, Eq. 2.7 has the form

$$2\,\mathrm{Im}\,\mathcal{M}(p_1 + p_2 \to p_1 + p_2) = \sum_\mathrm{X}\int \mathrm{d}\Pi_\mathrm{X}(2\pi)^4\delta^4(p_1 + p_2 - p_\mathrm{X})|\mathcal{M}(p_1 + p_2 \to \mathrm{X})|^2. \tag{2.8}$$

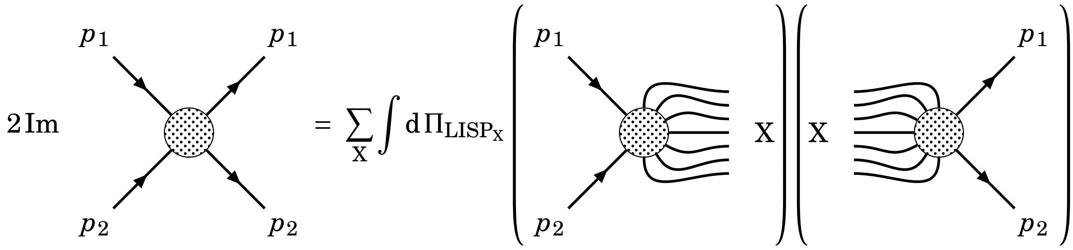


Figure 2.1: **The optical theorem.** The unitarity of the $\mathcal{S}$-matrix relates the imaginary part of forward scattering amplitude $p_1 + p_2 \to p_1 + p_2$ to the cross-section of $p_1 + p_2 \to \mathrm{X}$. The quantum numbers of the initial state $|i\rangle = |p_1, p_2\rangle$ and final state $|f\rangle = |p_1, p_2\rangle$ must be the same. This figure is a reproduction of Fig. 7.5 from [17, p. 231].

In addition, the identification $\mathrm{d}\Pi_{\mathrm{X}}(2\pi)^4\delta^4(p_1 + p_2 - p_{\mathrm{X}}) = \mathrm{d}\Pi_{\mathrm{LISP_X}}$ in Eq. C.4 yields a relation between the imaginary part of the forward amplitude $\mathcal{M}(p_1 + p_2 \to p_1 + p_2)$ and the cross-section of the scattering $p_1 + p_2 \to \mathrm{X}$

$$\begin{aligned} \mathrm{Im}\,\mathcal{M}(p_1 + p_2 \to p_1 + p_2) &= 2E_1E_2|\mathbf{v}_1 - \mathbf{v}_2| \sum_{\mathrm{X}} \sigma(p_1 + p_2 \to \mathrm{X}) \\ &= 2\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2} \sum_{\mathrm{X}} \sigma(p_1 + p_2 \to \mathrm{X}). \end{aligned} \tag{2.9}$$

The above equation is named the optical theorem. Note that if $\sigma(p_1 + p_2 \to \mathrm{X})$ is the cross-section associated to a tree diagram, then $\mathcal{M}(p_1 + p_2 \to p_1 + p_2)$ must represent a loop diagram.

2.2 Dispersion Relations

2.2.1 Derivation of Unsubstracted Dispersion Relations

Let $f(z)$ be an analytic function in the complex plane with no singularities aside a branch point at $\nu_0 \in \mathbb{R}$. Using the Cauchy theorem on the contour $\mathrm{C} = \mathrm{C}_\infty \cup \mathrm{C}_- \cup \mathrm{C}_{\nu_0} \cup \mathrm{C}_+$ of Fig. 2.2b, $f(z)$ can be expressed in its analytic domain through

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \oint_{\mathrm{C}} \mathrm{d}w \frac{f(w)}{w - z} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_+} \mathrm{d}w \frac{f(w)}{w - z} + \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_-} \mathrm{d}w \frac{f(w)}{w - z} \\ &\quad + \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_\infty} \mathrm{d}w \frac{f(w)}{w - z} + \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_{\nu_0}} \mathrm{d}w \frac{f(w)}{w - z}. \end{aligned} \tag{2.10}$$

As stated, ν_0 is only a branch point, it is not also a pole, thus the integration along C_{ν_0} is null. From the construction of C_+ and C_-, the integrals along these curves are

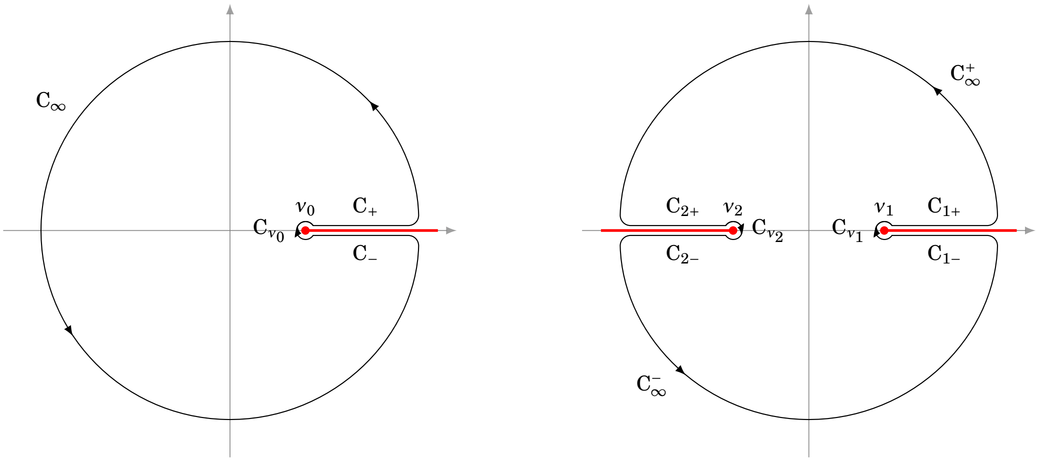
$$\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_+} \mathrm{d}w \frac{f(w)}{w - z} = \frac{1}{2\pi\mathrm{i}} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{f(x + \mathrm{i}\epsilon)}{x - z + \mathrm{i}\epsilon},$$

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_-} \mathrm{d}w \frac{f(w)}{w - z} = \frac{1}{2\pi\mathrm{i}} \int_{\infty}^{\nu_0} \mathrm{d}x \frac{f(x - \mathrm{i}\epsilon)}{x - z - \mathrm{i}\epsilon},$$

where the change of variable $w \to x \pm \mathrm{i}\epsilon$, with $\epsilon \to 0^+$ and $x \in \mathbb{R}$, is considered.

For simplicity, presume that $f(z) \to c$, c a constant, on the curve C_∞. Therefore, the integral along C_∞ is given by

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{C}_\infty} \mathrm{d}w \frac{f(w)}{w - z} = c.$$



(a) Integration contour with one branch

(b) Integration contour with two branches

Figure 2.2: **Integration contours for $f(z)$.** The function $f(z)$ has no singularities on and inside the contours, if stated otherwise. The integration path is taken counter-clockwise. (a) ν_0 is a branch point with no other singularities. The integration path consists of $C_+ \cup C_\infty \cup C_- \cup C_{\nu_0}$, (b) ν_1 and ν_2 are branch points with no other singularities. The integration path consists of $C_{1+} \cup C^+_\infty \cup C_{2+} \cup C_{\nu_2} \cup C_{2-} \cup C^-_\infty \cup C_{1-} \cup C_{\nu_1}$.

By substituting the integrals along C_ν, C_+, C_-, and C_∞ in Eq. 2.10, $f(z)$ reads as

$$\begin{aligned} f(z) &= c + \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \to 0^+} \int_{\nu_0}^{\infty} \mathrm{d}x \left[\frac{(x-z)[f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)]}{(x-z)^2+\epsilon^2} - \mathrm{i}\epsilon \frac{f(x+\mathrm{i}\epsilon)+f(x-\mathrm{i}\epsilon)}{(x-z)^2+\epsilon^2} \right] \\ &= c + \frac{1}{2\pi \mathrm{i}} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta(x)}{x-z}. \end{aligned} \tag{2.11}$$

The quantity $\Delta(x) = f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)$ is the discontinuity of $f(z)$ across the branch cut.

A useful generalization of the above result is when $f(z)$ has two branch points, ν_1 and ν_2, as in Fig. 2.2b. Suppose that $f(z) \to c$ on both C^+_∞ and C^-_∞. The same method as for one branch point can be employed. By making a change of variable for $x \to -x$ in the integrals for C_{2+} and C_{2-}, $f(z)$ is described by Eq. 2.12, where $\Delta_+(x) = f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)$ and $\Delta_-(x) = f(-x+\mathrm{i}\epsilon) - f(-x-\mathrm{i}\epsilon)$.

$$f(z) = c + \frac{1}{2\pi \mathrm{i}} \left[\int_{\nu_1}^{\infty} \mathrm{d}x \frac{\Delta_+(x)}{x-z} - \int_{-\nu_2}^{\infty} \mathrm{d}x \frac{\Delta_-(x)}{x+z} \right]. \tag{2.12}$$

Furthermore, if $f(z)$ has a reflection symmetry around the imaginary axis, then $\nu_1 = -\nu_2 = \nu_0$. In this context, two cases arise:

1. $f(z)$ is even around the imaginary axis, i.e. $f(-\nu+\mathrm{i}\theta) = f^*(\nu+\mathrm{i}\theta)$, thus $\Delta_+(x) = \Delta^*_-(x) = \Delta(x)$ and

$$\begin{aligned} f(z) &= c + \frac{1}{2\pi \mathrm{i}} \left[\int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta(x)}{x-z} - \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta^*(x)}{x+z} \right] \\ &= c + \frac{1}{2\pi \mathrm{i}} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{2x\mathrm{i}\,\mathrm{Im}\,\Delta(x) + 2z\,\mathrm{Re}\,\Delta(x)}{x^2-z^2}, \end{aligned} \tag{2.13}$$

2. $f(z)$ is odd around the imaginary axis, i.e. $f(-\nu+\mathrm{i}\theta) = -f^*(\nu+\mathrm{i}\theta)$, thus $\Delta_+(x) = -\Delta^*_-(x) = \Delta(x)$

and

$$
\begin{aligned}
f(z) &= c + \frac{1}{2\pi \mathrm{i}} \left[\int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta(x)}{x-z} + \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta^*(x)}{x+z} \right] \\
&= c + \frac{1}{2\pi \mathrm{i}} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{2x \operatorname{Re}\Delta(x) + 2z\mathrm{i} \operatorname{Im}\Delta(x)}{x^2 - z^2}.
\end{aligned}
\tag{2.14}
$$

In QFT, scattering amplitudes can have a region where they take real values on the real axis [19, p. 31]. This encourages the investigation of the function $f(z)$ when there is a domain $\mathscr{D}_0$ on which $f(z) \in \mathbb{R}$ for $z \in \mathbb{R} \cap \mathscr{D}_0$. This conditions, jointed with the analyticity of $f(z)$ implies that $f(z)$ must have a symmetry around the real axis, Schwarz reflection principle [20, p. 379], see Chapter 2.2.4.

Whenever $f(z)$ satisfies the Schwarz reflection principle, Chapter 2.2.4, the discontinuity $\Delta(x)$ is related to the imaginary or real part of the value that $f(z)$ takes just above the cut. In this thesis only the symmetry $f(z^*) = f^*(z)$ around the real axis is of concern.

Firstly consider $f(z)$ with one branch point, Fig. 2.2a. The symmetry around the real axis implies that $f(x+\mathrm{i}\epsilon) = f^*(x-\mathrm{i}\epsilon)$ and, in consequence, $\Delta(x) = 2\mathrm{i} \operatorname{Im} f(x+\mathrm{i}\epsilon)$. By substituting $\Delta(x)$ in Eq. 2.11, it follows that

$$
f(z) = c + \frac{1}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\operatorname{Im} f(x+\mathrm{i}\epsilon)}{x-z}. \tag{2.15}
$$

If $z = \nu + \mathrm{i}\theta$ extends a real quantity in the complex plane, then $\operatorname{Im} z = \theta > 0$ [20, p. 369]. Thus, to apprise the value of $f(z)$ for $z = \nu$ real, the limit $\theta \to 0^+$ must be taken:

$$
\begin{aligned}
f(\nu) &= \lim_{\theta \to 0^+} c + \frac{1}{\pi \mathrm{i}} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\operatorname{Im} f(x+\mathrm{i}\epsilon)}{x-z} \\
&= c + \frac{1}{\pi \mathrm{i}} \operatorname{Pr} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\operatorname{Im} f(x+\mathrm{i}\epsilon)}{x-\nu} + 2\mathrm{i}\pi \operatorname{Im} f(\nu),
\end{aligned}
$$

where Eq. 2.16 below, in the sense of a distribution, was used.

$$
\lim_{\theta \to 0^+} \frac{1}{x - \nu \mp \mathrm{i}\theta} = \operatorname{Pr} \frac{1}{x-\nu} \pm \mathrm{i}\pi\delta(x-\nu). \tag{2.16}
$$

The separation of $f(\nu)$ in imaginary and real part will be useful in the subsequent chapters of the paper,

$$
\begin{aligned}
\operatorname{Re} f(\nu) &= \operatorname{Re} c + \frac{1}{\pi} \operatorname{Pr} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\operatorname{Im} f(x+\mathrm{i}\epsilon)}{x-\nu}, \\
\operatorname{Im} f(\nu) &= \operatorname{Im} c + \operatorname{Im} f(\nu).
\end{aligned}
\tag{2.17}
$$

Note that $\operatorname{Im} c$ must be null, because the identity for $\operatorname{Im} f(\nu)$ must hold identically. Equation 2.17 represents the unsubstracted dispersion relation of a function that is even around the real axis and has one branch point on the real axis.

Secondly consider $f(z)$ to have two branch points $\nu_1 = -\nu_2 = \nu_0$, Fig. 2.2b. Based on the analysis done in this paper, only the case when $f(-z) = f(z)$ is considered, the function $f(z)$ has an even symmetry around both the imaginary and real axes.

If $f(-z) = f(z)$, then $\Delta(x) = 2\mathrm{i} \operatorname{Im} f(x+\mathrm{i}\epsilon)$, $\operatorname{Re}\Delta(x) = 0$, and $\operatorname{Im}\Delta(x) = 2 \operatorname{Im} f(x+\mathrm{i}\epsilon)$. Replacing $\Delta(x)$ in Eq. 2.13, $f(z)$ has the form

$$
f(z) = c + \frac{2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{x \operatorname{Im} f(x+\mathrm{i}\epsilon)}{x^2 - z^2}
$$

For a real $z = \nu + \mathrm{i}\theta$, $\theta \to 0^+$, the above relation becomes

$$
\begin{aligned}
f(\nu) &= c + \lim_{\theta\to 0^+} \frac{1}{2\pi\mathrm{i}} \left[\int_{\nu_0}^{\infty} \mathrm{d}x \frac{2\mathrm{i}\,\mathrm{Im}\, f(x+\mathrm{i}\epsilon)}{x-\nu} + \int_{\nu_0}^{\infty} \mathrm{d}x \frac{2\mathrm{i}\,\mathrm{Im}\, f(x+\mathrm{i}\epsilon)}{x+\nu} \right] \\
&= c + \frac{2}{\pi} \mathrm{Pr} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{x\,\mathrm{Im}\, f(x+\mathrm{i}\epsilon)}{x^2-\nu^2} \\
&\quad + \int_{\nu_0}^{\infty} \mathrm{d}x \delta(x-\nu)\mathrm{Im}\, f(x+\mathrm{i}\epsilon) - \int_{\nu_0}^{\infty} \mathrm{d}x \delta(x+\nu)\mathrm{Im}\, f(x+\mathrm{i}\epsilon).
\end{aligned}
$$

The first integral over $\delta(x-\nu)$ yields $\mathrm{Im}\, f(x+\mathrm{i}\epsilon)$ for $\nu \in (\nu_0, \infty)$. The second integral over $\delta(x-\nu)$ yields $\mathrm{Im}\, f(x+\mathrm{i}\epsilon)$ for $\nu \in (-\infty, -\nu_0)$, because the assumption $f(\nu+\mathrm{i}\theta) = f^*(-\nu+\mathrm{i}\theta)$ implies that $\mathrm{Im}\, f(x+\mathrm{i}\epsilon) = -\mathrm{Im}\, f(-x+\mathrm{i}\epsilon)$. Thus, the real and imaginary parts of $f(\nu)$ are

$$
\begin{aligned}
\mathrm{Re}\, f(\nu) &= \mathrm{Re}\, c + \frac{2}{\pi} \mathrm{Pr} \int_{\nu_0}^{\infty} \mathrm{d}x \frac{x\,\mathrm{Im}\, f(x+\mathrm{i}\epsilon)}{x^2-\nu^2}, \\
\mathrm{Im}\, f(\nu) &= \mathrm{Im}\, c + \mathrm{Im}\, f(\nu).
\end{aligned}
\tag{2.18}
$$

Note again that $\mathrm{Im}\, c$ has to be zero. Equation 2.18 is the unsubstracted dispersion relation of a function that is even and has two branch points on the real axis.

2.2.2 Sugawara-Kanazawa Theorem

A generalization of Eq. 2.12 is the Sugawara-Kanazawa theorem [16; 19, p. 32]. Amongst others, it allows the assessment of the constant c in Eq. 2.12 if the limits $f(\infty+\mathrm{i}\epsilon)$ and $f(\infty-\mathrm{i}\epsilon)$ on C_{1+} and C_{1-}, respectively, are known. This theorem plays an important part in the analysis carried in Chapter 3 and Chapter 4.

Theorem (Sugawara-Kanazawa)**.** *Let $f(z)$ be a complex function with the properties:*

1. *$f(z)$ is regular excepting two branch cuts on the real axis, as in Fig. 2.2b, and excepting a numerable set $\{w_i\}$ of poles on the real axis such that $\nu_2 < x_i < \nu_1$; the poles are not illustrated in Fig. 2.2b,*
2. *The limits $f(\infty+\mathrm{i}\epsilon)$ and $f(\infty-\mathrm{i}\epsilon)$, $\epsilon \to 0^+$, exist and are finite on the cuts C_{1+} and C_{1-}, respectively,*
3. *The limits $f(-\infty+\mathrm{i}\epsilon)$ and $f(-\infty-\mathrm{i}\epsilon)$ tend each to monotone functions along the cuts C_{2+} and C_{2-}, respectively,*
4. *There exists $n \in \mathbb{N}$ such that $|f(z)| < |z|^n$ as $|z| \to \infty$.*

Under these assumptions,

$$
\begin{aligned}
\lim_{|z|\to\infty} f(z) &= f(\infty+\mathrm{i}\epsilon) \quad \text{for any } z \in \mathrm{C}_{\infty}^{+}, \\
\lim_{|z|\to\infty} f(z) &= f(\infty-\mathrm{i}\epsilon) \quad \text{for any } z \in \mathrm{C}_{\infty}^{-},
\end{aligned}
\tag{2.19}
$$

and

$$
\begin{aligned}
f(z) &= \sum_i \frac{\mathrm{Res}[f(w_i)]}{z-w_i} + \frac{1}{2}[f(\infty+\mathrm{i}\epsilon) + f(\infty-\mathrm{i}\epsilon)] \\
&\quad + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\nu_2} \mathrm{d}x \frac{f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)}{x-z} \\
&\quad + \frac{1}{2\pi\mathrm{i}} \int_{\nu_1}^{\infty} \mathrm{d}x \frac{f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)}{x-z},
\end{aligned}
\tag{2.20}
$$

where $\operatorname{Res}[f(w_i)]$ *is the residue of* $f(z)$ *at* w_i.

2.2.3 Derivation of Once-Substracted Dispersion Relations

Chapter 3 and Appendix E.1, where the Baldin sum rule (Baldin-SR) is employed, makes use of the so called once-substracted dispersion relations. The Baldin-SR assumes a function $f(z)$ that: (i) is even around both the real and the imaginary axes; (ii) has two branch cuts as in Fig. 2.2b, with the branch points $\nu_0 = \nu_1 = -\nu_2$; (iii) takes a defined value $f(x_1)$ at some point x_1 on the real axis, with $x_1 \in (-\nu_0, \nu_0)$. The function $f(z)$ can be also described using a dispersion relation.

To construct a dispersion relation for $f(z)$, consider the function

$$g(z) = \frac{f(z) - f(x_1)}{z - x_1}. \tag{2.21}$$

The function $g(z)$ is analytic in x_1 because

$$g(x_1) = \lim_{z \to x_1} \frac{f(z) - f(x_1)}{z - x_1} = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\Big|_{z=x_1}.$$

Apply Eq. 2.12 to $g(z)$, with the outcome

$$g(z) = \frac{f(z) - f(x_1)}{z - x_1} = \frac{1}{2\pi\mathrm{i}}\left[\int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta_+(x)}{(x-z)(x-x_1)} + \int_{\nu_0}^{\infty} \mathrm{d}x \frac{\Delta_-(x)}{(x+z)(x+x_1)}\right]. \tag{2.22}$$

To maintain the calculations simple, the constant c is considered to be null, e.g. $c = 0$ if $g(z) \to 0$ for $|z| \to \infty$, $z \in \mathrm{C}^+_\infty \cup \mathrm{C}^-_\infty$. The quantities $\Delta_+(x)$ and $\Delta_+(x)$ are still given by the relations: $\Delta_+(x) = f(x+\mathrm{i}\epsilon) - f(x-\mathrm{i}\epsilon)$ and $\Delta_-(x) = f(-x+\mathrm{i}\epsilon) - f(-x-\mathrm{i}\epsilon)$, where $\epsilon \to 0^+$. Note that in comparison with Eq. 2.12, there is a plus sign between the two integrals of the above equation. This sign comes from the additional term $(x - x_1) \to -(x + x_1)$ in the denominator when the change of variable $x \to -x$ is made.

Because $f(z)$ is even around the imaginary axis, then $\Delta_+(x) = \Delta^*_-(x) = \Delta(x)$. With some algebra, Eq. 2.22 can be rewritten as

$$g(z) = \frac{f(z) - f(x_1)}{z - x_1} = \frac{1}{2\pi\mathrm{i}}\int_{\nu_0}^{\infty} \mathrm{d}x \frac{2x\mathrm{i}(z + x_1)\operatorname{Im}\Delta(x) + 2\left(x^2 + zx_1\right)\operatorname{Re}\Delta(x)}{(x^2 - z^2)(x^2 - x_1^2)}. \tag{2.23}$$

As $f(z)$ is also even around the real axis, the Schwarz principle implies that $\Delta(x) = 2\mathrm{i}\operatorname{Im} f(x+\mathrm{i}\epsilon)$. By replacing $\operatorname{Im}\Delta(x) = 2\operatorname{Im} f(x+\mathrm{i}\epsilon)$ and $\operatorname{Re}\Delta(x) = 0$, and by taking the limit $z = \nu + \mathrm{i}\theta \to \nu + \mathrm{i}0^+$, from Eq. 2.23 follows the once-substracted dispersion relation for $f(\nu)$:

$$\operatorname{Re} f(\nu) = f(x_1) + \frac{2}{\pi}(\nu^2 - x_1^2)\Pr\int_{\nu_0}^{\infty} \mathrm{d}x \frac{x \operatorname{Im} f(x+\mathrm{i}\epsilon)}{\left(x^2 - \nu^2\right)\left(x^2 - x_1^2\right)}, \tag{2.24}$$

$$\operatorname{Im} f(\nu) = \operatorname{Im} f(\nu).$$

2.2.4 Schwarz Reflection Principle

Theorem (Schwarz reflection principle). *Let $f(z)$ be an analytic function on a domain $\mathscr{D}$ of the complex plane. Suppose that there exists a subdomain $\mathscr{D}_0 \in \mathscr{D}$ on which $f(z) \in \mathbb{R}$ for $z \in \mathbb{R} \cap \mathscr{D}_0$, or $f(z) \in \mathrm{i}\mathbb{R}$ for $z \in \mathbb{R} \cap \mathscr{D}_0$. Then, $f(z)$ has a symmetry around the real axis, or around the imaginary axis, respectively.*

Proof. The Schwarz principle can be proved by making a Taylor expansion around a point $x \in \mathscr{D}_0$. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(z - x)^n,$$

for some $z \in \mathscr{D}_0$ and $x \in \mathscr{D}_0 \cap \mathbb{R}$. If $f(x) \in \mathbb{R}$, then any derivative of $f(x)$ are in $\mathbb{R}$, thus

$$\begin{aligned} f(z^*) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(z^* - x)^n \\ &= \left[\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(z - x)^n \right]^* \\ &= f^*(z). \end{aligned}$$

For the case when $f(x) \in \mathrm{i}\mathbb{R}$, the derivatives of $f(x)$ belong to $\mathrm{i}\mathbb{R}$ and

$$\begin{aligned} f(z^*) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(z^* - x)^n \\ &= -\left[\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(z - x)^n \right]^* \\ &= -f^*(z). \end{aligned}$$

Analytic continuity of $f(z)$ from the domain $\mathscr{D}_0$ to the complex plane guarantees that the Schwarz reflection principle on the domain $\mathscr{D}$. □

Chapter 3

Unsubtracted Sum Rule for Longitudinal Polarization of the Photon

The analysis conducted in this paper focuses on the photo-disintegration process $\gamma + \mathrm{P} \to \mathrm{X}$, and the corresponding forward double virtual Compton scattering (FVVCS) $\gamma + \mathrm{P} \to \gamma + \mathrm{P}$ in leading order QED. The symbol γ represents a photon, P can be a vector or a scalar neutral particle, and X is a fermion pair e^+e^- or a charged scalar pair $\pi^\dagger\pi$. The attention will be mainly concentrated on the Lagrangians listed in Table 3.1. For most of the entries in Table 3.1, the photon field A_μ does not interact directly with the particle P, and the interaction involves a fermion field ψ or a charged scalar field π. All the Lagrangians have electro-magnetic invariance. The relevant Feynman rules used in this work are listed in Appendix B.

3.1 Cross-Sections at Tree Level

In leading order, the photo-disintegration $\gamma + \mathrm{P} \to \mathrm{X}$ is given by Feynman tree diagrams. The kinematics of the process is defined via the variables: (i) q is the four-momentum of the photon γ and $Q^2 = -q^2$ is the energy-momentum transfer of the photon, (ii) p is the four momentum of the particle P and M is the mass of the particle P, with $M^2 = p^2$, (iii) $\nu = \frac{p.q}{M}$ is the fractional energy of the photon, (iv) m is the mass of the particles in the pair X.

$\sigma_{\mathrm{T}}(\nu, Q^2)$ stands for the photo-absorption cross-section averaged over the transversal polarizations of the photon. $\sigma_{\mathrm{L}}(\nu, Q^2)$ is the photo-disintegration cross-section when the photon is longitudinal polarized. If applicable, the polarizations of the particle P and the spins of the pair X are also averaged when computing $\sigma_{\mathrm{T}}(\nu, Q^2)$ or $\sigma_{\mathrm{L}}(\nu, Q^2)$. For an overview of kinematics and cross-section calculations, see Appendix C.1 and C.2.

3.2 Forward Double Virtual Compton Scattering

The FVVCS $\gamma + \mathrm{P} \to \gamma + \mathrm{P}$ is characterized by the Mandelstam variable $t = 0$, i.e. incoming momenta for each particle are equal to outgoing momenta, respectively, see Fig. 3.1. If FVVCS exhibits electro-magnetic gauge invariance, then the matrix element $\mathcal{M}^{\mu\nu}$, possibly averaged over the polarizations of the particle P, has the gauge invariant structure [21]

$$\mathcal{M}^{\mu\nu}(\nu, Q^2) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2(\nu, Q^2). \tag{3.1}$$

$T_1(\nu, Q^2)$ and $T_2(\nu, Q^2)$ are scalar functions that depend only on the variables ν and Q^2. For a short discussion about FVVCS refer to Appendix D.2.

Table 3.1: **List of Lagrangians analyzed in the current paper**. m is the mass of the fermion field ψ, or charged scalar field π, depending on the context. M is the mass of the scalar field ϕ, or of the massive vector field B_μ, depending on the context. A_μ is the photon field. e is the electric coupling. g is the coupling of ϕ or B_μ to ψ or π, also context dependent. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

#	Lagrangian
1.	$\mathscr{L}_1 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
2.	$\mathscr{L}_2 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + \mathrm{i}g\bar{\psi}\gamma_5\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
3.	$\mathscr{L}_3 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
4.	$\mathscr{L}_4 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\psi B_\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
5.	$\mathscr{L}_5 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
6.	$\mathscr{L}_6 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\sigma^{\mu\nu}\psi G_{\mu\nu} - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$
7.	$\mathscr{L}_7 = \left[(\partial_\mu + \mathrm{i}eA_\mu)\pi\right]^\dagger[(\partial^\mu + \mathrm{i}eA^\mu)\pi] + g\pi^\dagger\pi\phi - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2$
8.	$\mathscr{L}_8 = \left[(\partial_\mu + \mathrm{i}eA_\mu + \mathrm{i}gB_\mu)\pi\right]^\dagger[(\partial^\mu + \mathrm{i}eA^\mu + \mathrm{i}gB^\mu)\pi] - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu$

The transversal amplitude of $\mathscr{M}^{\mu\nu}(\nu,Q^2)$ is the function $T_1(\nu,Q^2)$. The longitudinal amplitude of $\mathscr{M}^{\mu\nu}(\nu,Q^2)$ has the form

$$T_\mathrm{L}(\nu,Q^2) = \epsilon_{\mathrm{L}\mu}\epsilon_{\mathrm{L}\nu}\mathscr{M}^{\mu\nu}(\nu,Q^2) = -T_1(\nu,Q^2) + \frac{\nu^2+Q^2}{Q^2}T_2(\nu,Q^2), \tag{3.2}$$

where ϵ_L is the longitudinal polarization of the photon. FVVCS has crossing symmetry under the exchange of the photons, thus $\mathscr{M}^{\mu\nu}(\nu,Q^2)$, $T_1(\nu,Q^2)$, and $T_\mathrm{L}(\nu,Q^2)$ are even functions in the variable ν.

3.3 Dispersion Relations and Sum Rules

A consequence of causality is the analyticity of the scattering amplitude in the complex energy plane, up to branch cuts on the real axis, which have physical interpretation. In the FVVCS $\gamma + \mathrm{P} \to \gamma + \mathrm{P}$, this implies the analyticity of the matrix element $\mathscr{M}^{\mu,\nu}(\nu,Q^2)$ in the energy complex plane ν, excepting branch cuts along the real axis. Because $\mathscr{M}^{\mu,\nu}(\nu,Q^2)$ is an even function of ν, there are two branch cuts on the real axis: $(-\infty,\nu_0]$ and $[\nu_0,\infty)$, where ν_0 is the threshold of particle photo-production. Thus, $T_1(\nu,Q^2)$ and $T_\mathrm{L}(\nu,Q^2)$ are also even analytic functions in ν with the branch cuts $(-\infty,\nu_0]$ and $[\nu_0,\infty)$.

Analyticity paired with the optical theorem, see Chapter 2.1, enables a set of dispersion relations for $T_1(\nu,Q^2)$ and $T_\mathrm{L}(\nu,Q^2)$. A short mathematical introduction to dispersion relations is available in Chapter 2.2. Crossing and convergence properties impose a once-substracted dispersion relation for $T_1(\nu,Q^2)$ under the form

$$T_1(\nu,Q^2) = T_1(0,Q^2) + \frac{4\nu^2 M}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \frac{\sqrt{\nu'^2+Q^2}}{\nu'(\nu'^2-\nu^2)}\sigma_\mathrm{T}(\nu',Q^2). \tag{3.3}$$

Based on the above equation, the low energy theorem [3; 4] and the interplay between $T_1(\nu,Q^2)$ and $T_2(\nu,Q^2)$ in the limit $Q^2 \to 0$, Baldin [5] linked the electric $\alpha_{\mathrm{E}1}$ and magnetic $\beta_{\mathrm{M}1}$ dipole polarizabilities of a particle to the photo-absorption cross-section $\sigma(\nu) = \sigma_\mathrm{T}(\nu,0)$ through

$$\alpha_{\mathrm{E}1} + \beta_{\mathrm{M}1} = \frac{1}{Q^2}T_2(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{4M}{\pi}\int_{\nu_0}^{\infty}\mathrm{d}\nu' \frac{\sigma(\nu')}{\nu'^2} \qquad \text{Baldin Sum Rule (Baldin-SR).} \tag{3.4}$$

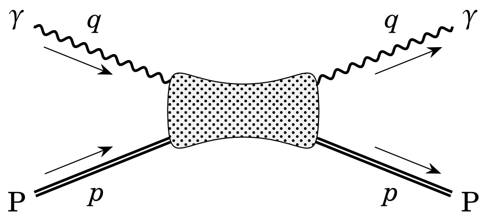


Figure 3.1: **General diagram of the FVVC process**. The incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the momentum of the photon γ, p is the momentum of the particle P.

Appendix E.1 presents a derivation of Eq. 3.4 for a neutral particle.

The longitudinal amplitude $T_{\rm L}(\nu,Q^2)$ can be written using an unsubstracted dispersion relation

$$\frac{1}{Q^2}T_{\rm L}(\nu,Q^2) = \frac{4M^2}{\pi}\int\limits_{\nu_0}^{\infty}\mathrm{d}\nu'\,\frac{\nu'\sqrt{\nu'^2+Q^2}}{\nu'^2-\nu^2}\frac{\sigma_{\rm L}(\nu',Q^2)}{Q^2}, \tag{3.5}$$

in the hope that $\frac{1}{Q^2}T_{\rm L}(\nu,Q^2)$ tends to 0, at least in the limit $\nu\to 0$ and $Q^2\to 0$. With this desiderate, the following sum rule was proposed in [12; 13]

$$\alpha_{\rm E1} = \frac{1}{Q^2}T_{\rm L}(0,Q^2)\Big|_{Q^2\to 0} = \frac{4M}{\pi}\int\limits_{\nu_0}^{\infty}\mathrm{d}\nu'\,\frac{\sigma_{\rm L}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0}. \tag{3.6}$$

A derivation of the above relation is accessible in Appendix E.3. In [15] it was proven that Eq. 3.6 holds up to a constant given by $\frac{T_{\rm L}(\nu,Q^2)}{Q^2}\big|^{\nu\to\infty}_{Q^2\to 0}$.

The current work establishes the following equation

$$\frac{1}{Q^2}T_{\rm L}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} - \frac{1}{Q^2}T_{\rm L}(\nu,Q^2)\Big|_{\substack{\nu\to\infty\\ Q^2\to 0}} = \frac{4M}{\pi}\int\limits_{\nu_0}^{\infty}\mathrm{d}\nu'\,\frac{\sigma_{\rm L}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0} \tag{3.7}$$

as an improved variation of Eq. 3.6. This variation is based on the hypotheses of the Sugawara-Kanazawa theorem (SK) [16], the statement of the theorem in Chapter 2.2.2. In the rest of the paper, Eq. 3.7 designates the longitudinal-sum rule (LSR) and the following notation is used

$$\Delta_{\rm L} = \frac{T_{\rm L}(\nu,Q^2)}{Q^2}\Big|_{\substack{\nu\to\infty\\ Q^2\to 0}}.$$

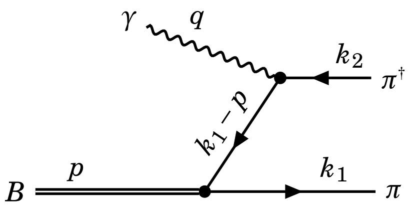
3.4 Investigation of the Longitudinal Sum Rule and Baldin Sum Rule by Using the Photo-Disintegration of a Vector Particle as Example

For exemplification, consider the Lagrangian

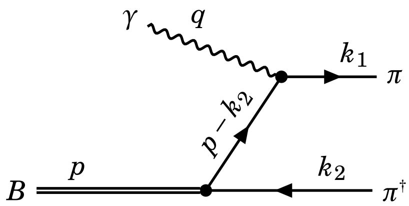
$$\mathscr{L}_8 = \left[(\partial_\mu + \mathrm{i}eA_\mu + \mathrm{i}gB_\mu)\pi\right]^\dagger\left[(\partial^\mu + \mathrm{i}eA^\mu + \mathrm{i}gB^\mu)\pi\right] - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2B_\mu B^\mu.$$

Photo-disintegration is given by $\gamma + B \to \pi + \pi^\dagger$, with the tree level Feynman diagrams in Fig. 3.2. The associated FVVCS process $\gamma + B \to \gamma + B$ has the Feynman diagrams in Fig. 3.3. The schematic representation of photo-disintegration and FVVCS for all the other cases is a subset of the diagrams in Figs. 3.2 and 3.3, respectively.

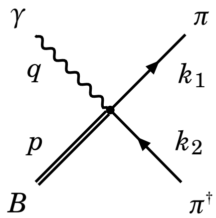
The computation of $\sigma_{\rm T}(\nu,Q^2)$ and $\sigma_{\rm L}(\nu,Q^2)$ are straightforward. Appendix D.1.1 presents a detailed calculation of $\sigma_{\rm T}(\nu,0)$ for $\mathscr{L}_2$, followed in Appendix D.1.2 by a cross-check which exploits the optical theorem. Appendix D.1.3 provides the full expressions of $\sigma_{\rm T}(\nu,Q^2)$ and $\sigma_{\rm L}(\nu,Q^2)$ of the



(a) Photon interaction with charged scalar particle



(b) Photon interaction with charged scalar anti-particle



(c) Four-point interaction of photon and massive vector with charged scalar particles

Figure 3.2: **Tree level diagram of the photo-disintegration process for $\mathscr{L}_8$.** The four-momenta of the particles are: q for the photon γ, p for the massive vector, k_1 for the charged scalar particle π, and k_2 for the charged scalar anti-particle $\pi^\dagger$.

cases $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_4$, and $\mathscr{L}_5$. The expressions for $\sigma_{\mathrm{T}}(\nu,Q^2)|_{Q^2\to 0}$ and $\frac{1}{Q^2}\sigma_{\mathrm{L}}(\nu,Q^2)|_{Q^2\to 0}$ corresponding to the interactions discussed in this paper are listed in Table D.1.

For $\mathscr{L}_8$, the quantities required in the RHS of Eqs. 3.7 and 3.4, the evaluation of LSR and Baldin-SR, are

$$\frac{4M}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \frac{\sigma_{\mathrm{L}}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0} = \frac{e^2g^2\left(4\rho^4-10\rho^2+15\right)}{288\pi^2m^2\rho^2\left(1-\rho^2\right)} - \frac{e^2g^2\left(4\left(\rho^2-7\right)\rho^2+15\right)}{288\pi^2m^2\rho^3\left(1-\rho^2\right)^{3/2}}\arcsin(\rho), \tag{3.8}$$

$$\frac{4M}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \frac{\sigma_{\mathrm{T}}(\nu',Q^2)}{\nu'^2}\Big|_{Q^2\to 0} = \frac{e^2g^2\left(13-10\rho^2\right)}{96\pi^2m^2\rho^2\left(1-\rho^2\right)} - \frac{e^2g^2\left(8\left(\rho^2-7\right)\rho^2+39\right)}{288\pi^2m^2\rho^3\left(1-\rho^2\right)^{3/2}}\arcsin(\rho), \tag{3.9}$$

where $\rho = \frac{M}{2m}$. As mentioned in Chapter 3.3, the LHS of Eq. 3.7 and Eq. 3.4 are related to the FVVCS amplitude $\mathscr{M}^{\mu\nu}$ of the diagrams in Fig. 3.3. The evaluation of these relations for $\mathscr{L}_8$, when $\nu \to 0$ and $Q^2 \to 0$, yields

$$\frac{T_{\mathrm{L}}(\nu,Q^2)}{Q^2}\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{e^2g^2\left(4\rho^4-6\rho^2+5\right)}{96\pi^2m^2\rho^2\left(1-\rho^2\right)} - \frac{e^2g^2\left(4\left(\rho^2-7\right)\rho^2+15\right)}{288\pi^2m^2\rho^3\left(1-\rho^2\right)^{3/2}}\arcsin(\rho), \tag{3.10}$$

$$\frac{T_2(\nu,Q^2)}{Q^2}\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{e^2g^2\left(13-10\rho^2\right)}{96\pi^2m^2\rho^2\left(1-\rho^2\right)} - \frac{e^2g^2\left(8\left(\rho^2-7\right)\rho^2+39\right)}{288\pi^2m^2\rho^3\left(1-\rho^2\right)^{3/2}}\arcsin(\rho), \tag{3.11}$$

$$\Delta_{\mathrm{L}} = \lim_{\substack{\nu\to\infty\\ Q^2\to 0}} \frac{1}{Q^2}T_{\mathrm{L}}(\nu,Q^2) = -\frac{e^2g^2}{36\pi^2m^2}. \tag{3.12}$$

The computations of T_2 and T_{L} are more involved than the ones for the cross-sections due to the presence of Feynman loop-integrals. In consequence, an algorithm for reducing tensor one-loop integrals to scalar loop-integrals and the evaluation of the latter was implemented in MATHEMATICA[1]. Appendix F.2 contains details about the design of the algorithm, while the

[1] In the limit $\nu \to 0$ and $Q^2 \to 0$, the calculations were double cross-checked by using a direct computation of the tensor one-loop integrals, and X-PACKAGE [22]. For the limit $\nu \to \infty$, both methods are inadequate.

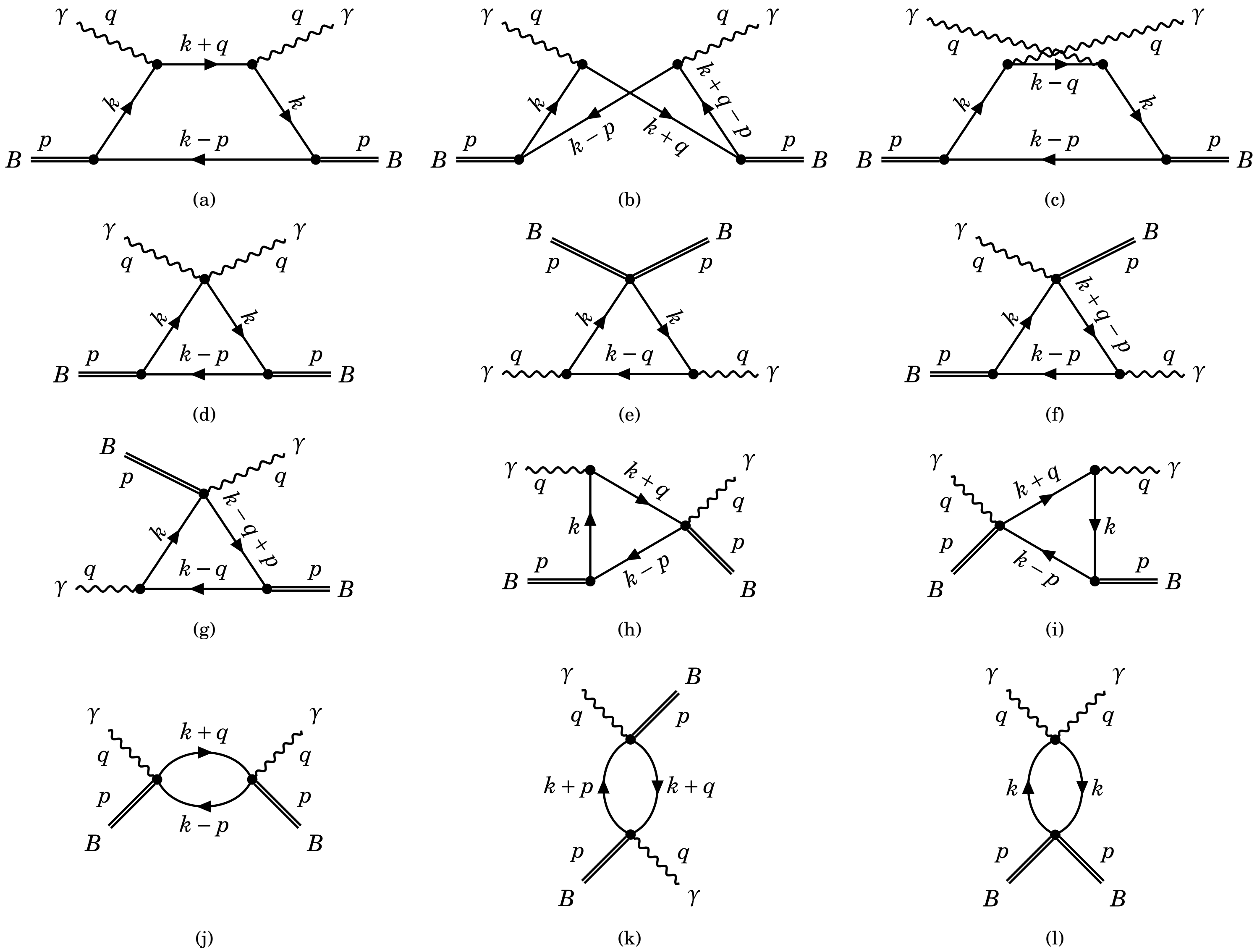


Figure 3.3: **Feynman diagrams of the FVVCS process between a photon and a neutral vector particle. The FVVCS process is governed by the Lagrangian $\mathscr{L}_8$.** Leading FVVCS is achieved through a one-loop diagram and the process involves the charged scalar particles π, $\pi^\dagger$. Incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the momentum of the photon γ, p is the momentum of the neutral vector particle B, and k is the unbounded momentum of the one-loop. The crossed diagrams are not displayed.

decomposition of $\mathscr{M}^{\mu\nu}$ in scalar one-loop integrals is presented in Appendix F.3 for $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_4$, and $\mathscr{L}_5$.

Equations 3.9 and 3.11 confirm the Baldin-SR given by Eq. 3.4. The LSR in the from of Eq. 3.7 is also confirmed based on Eqs. 3.8, 3.10, and 3.12. This result for the LSR is in agreement with the SK theorem, yet it also suggests that the LSR should be rewritten in a subtracted from.

A closer study of the FVVCS diagrams in Fig. 3.3 indicates that a different weight of the diagrams Figs. 3.3e and 3.3l in the entire matrix element would actually make the LSR hold exactly, i.e. $\Delta_\mathrm{L} = 0$. Adding the following term

$$\mathscr{L}'_{8,\mathrm{int}} = -\frac{2}{3} g^2 \pi^\dagger \pi B_\mu B^\mu \tag{3.13}$$

to $\mathscr{L}_8$, the new Lagrangian

$$\mathscr{L}'_8 = \left[(\partial_\mu + ieA_\mu + igB_\mu)\pi\right]^\dagger \left[(\partial^\mu + ieA^\mu + igB^\mu)\pi\right] - \frac{2}{3} g^2 \pi^\dagger \pi B_\mu B^\mu$$
$$- m\pi^\dagger \pi - \frac{1}{4} F_\mu F^\mu - \frac{1}{4} G_\mu G^\mu + \frac{1}{2} M^2 B_\mu B^\mu$$

yields a LSR with $\Delta_\mathrm{L} = 0$. However, $\mathscr{L}'_8$ no longer has gauge invariance in the massive vector field B. Figure 3.4 depicts the FVVCS Feynman diagrams corresponding to $\mathscr{L}'_{8,\mathrm{int}} \sim g\pi^\dagger \pi B_\mu B^\mu$, implicitly appended with the interaction between the field A_μ and the field π. There is no possible cut to reduce the diagrams in Fig. 3.4 to the tree level diagrams in Fig. 3.2. Thus, σ_L and σ_T of $\mathscr{L}'_8$ are identical to those of $\mathscr{L}_8$. A notable property of the diagrams in Fig. 3.4 is the electro-magnetic gauge-invariance.

The amplitude $T_{L8,\mathrm{int}}(\nu, Q^2)$ for Fig. 3.4 does not depend on ν

$$T_{L8,\mathrm{int}}(\nu, Q^2) = \frac{e^2 g^2}{3\pi^2} \left[-1 + \frac{\sqrt{4m^2 + Q^2}}{Q} \operatorname{arcsinh}\left(\frac{Q}{2m}\right) \right]. \tag{3.14}$$

This is also valid for the part in $T_\mathrm{L}(\nu, Q^2)$ associated to Figs. 3.3e and 3.3l. Therefore, in the limit $Q^2 \to 0$ and any ν, the LSR Eq. 3.7 holds for $\mathscr{L}_8$ up to $\Delta_\mathrm{L} = -\frac{e^2 g^2}{36\pi^2 m^2}$, and for $\mathscr{L}'_8$ exactly. This was expressly checked, see Eq. E.22 for the closed form of $\frac{1}{Q^2} T_\mathrm{L}(\nu, Q^2)|_{Q^2 \to 0}$ in the $\mathscr{L}_8$ case.

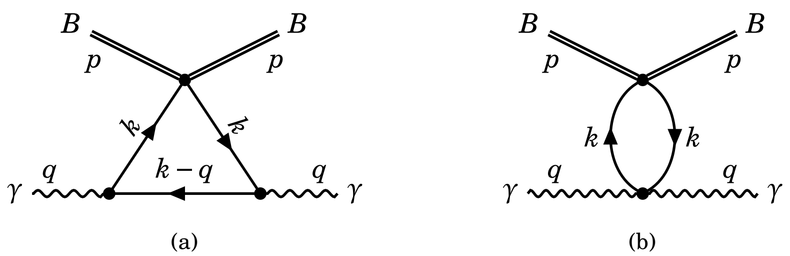


Figure 3.4: **FVVCS Feynman diagrams of $\mathscr{L}'_{8,\mathrm{int}}$ with the additional interaction of the photon with the charged scalar field**. Incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the photon momentum, p is the momentum of the neutral vector particle B, and k is the unbounded momentum of the one-loop. The crossed diagrams are not displayed.

3.5 Results and Discussions of the Longitudinal Sum Rule

The procedure to compute the Baldin-SR and LSR of the photo-disintegration process for the Lagrangians in Table 3.1 is similar to the one presented in Chapter 3.4. In leading order, the Baldin-SR holds for the photo-disintegration process for all the Lagrangians under study. In a sense, Baldin-SR was used during the calculations as a cross-check. The results for the Baldin-SR corresponding to Table 3.1 are listed in Table E.1.

Table 3.2: $\Delta_{\mathrm{L}} = \frac{T_{\mathrm{L}}(\nu,Q^2)}{Q^2}|_{Q^2\to 0}^{\nu\to\infty}$ **for the interactions considered in the present work**. m is the mass of the fermion ψ, or charged particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, context dependent. ϕ is a scalar field, B_μ is a vector field. The last column represents the dimension of the interaction Lagrangian without counting the dimension of the couplings.

#	$\mathscr{L}_{\mathrm{int}}$	Δ_{L}	$[\mathscr{L}_{\mathrm{int}}]_{\mathrm{fields}}$
1.	$\mathscr{L}_{1,\mathrm{int}} = g\bar{\psi}\psi\phi$	$-\frac{e^2g^2}{6\pi^2m^2}$	4
2.	$\mathscr{L}_{2,\mathrm{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi$	$-\frac{e^2g^2}{6\pi^2m^2}$	4
3.	$\mathscr{L}_{3,\mathrm{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi$	0	5
4.	$\mathscr{L}_{4,\mathrm{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu$	$-\frac{e^2g^2}{9\pi^2m^2}$	4
5.	$\mathscr{L}_{5,\mathrm{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu$	$-\frac{e^2g^2}{9\pi^2m^2}$	4
6.	$\mathscr{L}_{6,\mathrm{int}} = g\bar{\psi}\sigma_{\mu\nu}\psi(\partial^\mu B^\nu - \partial^\nu B^\mu)$	$-\frac{2e^2g^2M^2}{9\pi^2m^2}$	5
7.	$\mathscr{L}_{7,\mathrm{int}} = g\pi^\dagger\pi\phi$	0	3
8.	$\mathscr{L}_{8,\mathrm{int}} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu + g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	$-\frac{e^2g^2}{36\pi^2m^2}$	4
a.	$\mathscr{L}_{\mathrm{a,int}} = g\bar{\psi}\psi\phi^2$	$+\frac{e^2g}{3\pi^2m}$	5
b .	$\mathscr{L}_{\mathrm{b,int}} = g\bar{\psi}\psi B_\mu B^\mu$	$-\frac{e^2g}{3\pi^2m}$	5
c.	$\mathscr{L}_{\mathrm{c,int}} = g\pi^\dagger\pi\phi^2$	$+\frac{e^2g}{24\pi^2m^2}$	4
d.	$\mathscr{L}_{\mathrm{d,int}} = g\pi^\dagger\pi B_\mu B^\mu$	$-\frac{e^2g}{24\pi^2m^2}$	4
3'.	$\mathscr{L}'_{3,\mathrm{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi + \frac{1}{2m}g^2\bar{\psi}\psi\phi^2$	0	$4 \oslash 5$
4'.	$\mathscr{L}'_{4,\mathrm{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu - \frac{1}{3m}g^2\bar{\psi}\psi B_\mu B^\mu$	0	$4 \oslash 5$
8'.	$\mathscr{L}'_{8,\mathrm{int}} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu + \frac{1}{3}g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	0	4

A modified version of the LSR is proposed in Eq. 3.7. This relation holds for the Lagrangians in Table 3.1 in the case of photo-disintegration process within the framework of leading order QED. This is also true for any ν when $Q^2 \to 0$. The non-primed arabic entries in Table 3.2 highlight the values of Δ_{L} for the studied interactions. Table E.2 is a detailed version of the LSR results in Table 3.2

As pointed in Table 3.2, the interaction $\mathscr{L}_{7,\mathrm{int}} = g\pi^\dagger\pi\phi$ yields a null value for Δ_{L}. The interaction Lagrangian $\mathscr{L}_{7,\mathrm{int}}$ is an example of a renormalizable theory that does not need modifications in order to meet the LSR exactly. The Feynman diagrams of the FVVCS are similar to Figs. 3.3a, 3.3b, 3.3c, and 3.3e, up to the replacement $B \to \phi$.

The interaction Lagrangian $\mathscr{L}_{3,\mathrm{int}}$ is an example of a non-renormalizable theory that satisfies the LSR exactly. The FVVCS diagrams are given by Figs. 3.3a, 3.3b, and 3.3c, with the modification $B \to \phi$. By changing the coupling $g \to \frac{g^2}{2m}$, $\mathscr{L}_{3,\mathrm{int}}$ is equivalent to $\mathscr{L}'_{3,\mathrm{int}}$ to first order through a chiral field redefinition of the field ψ [23]. However, $\mathscr{L}'_{3,\mathrm{int}}$ has the FVVCS diagrams of Figs. 3.3a, 3.3b, 3.3c, and 3.3e, with $B \to \phi$. The new triangle diagram Fig. 3.3e is in a one-to-one correspondence to the term $\frac{1}{2m}g^2\bar{\psi}\psi\phi^2$.

Given the Lagrangians in Table 3.1, a four-point UV completion can be derived from the above examples in order to drive the quantity Δ_L to zero, at least in leading order QED. The lettered entries of Table 3.2 are illustrations of such UV completions, or prescription, for $\mathscr{L}_1$–$\mathscr{L}_8$. The Lagrangians $\mathscr{L}_{\mathrm{a,init}}$, $\mathscr{L}_{\mathrm{b,init}}$, and $\mathscr{L}_{\mathrm{c,init}}$ generate a FVVCS diagram as Fig. 3.4a, and $\mathscr{L}_{\mathrm{d,init}}$ generates Figs. 3.4a and 3.4b. With a suitable selection of the coupling constants, the primed arabics Lagrangians in Table 3.2 can be set up, which yield $\Delta_\mathrm{L} = 0$.

A further reason that mitigates for the four-point UV completion is the form of the scalar one-loop decomposition of the FVVCS matrix element $\mathscr{M}^{\mu\nu}$ in the cases $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_4$, and $\mathscr{L}_5$, see Appendix F.3. For these cases, the only scalar-integral that gives rise to the constant Δ_L, for any ν when $Q^2 \to 0$, is

$$B(q^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2 - m^2\right]\left[(k-q)^2 - m^2\right]}. \tag{3.15}$$

Nevertheless, $B(q^2)$ is essential for the electro-magnetic gauge invariance. $B(q^2)$ matches a diagram similar to Fig. 3.4b.

It might not be physical that the scattering amplitude of a photon with an infinite energy should depend on the virtuality of the photon, see Eqs. 3.14 and 3.15, and Table E.3. If it is assumed that $T_\mathrm{L}(\nu, Q^2)|_{\nu\to\infty} \in \{0, \infty\}$, then $\Delta_\mathrm{L} = 0$. In consequence, $\Delta_\mathrm{L} \neq 0$ could be interpreted as an artifact of the theory, at least in leading order QED, and the UV completion could be a way to remove this artefact.

The four-point UV completion suggests the introduction of a Nambu-Jona-Lasinio interaction $\mathscr{L}_{\mathrm{NJL}} \sim (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2$ such that $\Delta_\mathrm{L} = 0$ for the result in [15], provided the interaction constant is suitable chosen. However, $\mathscr{L}_{\mathrm{NJL}}$ is a non-renormalizable interaction. The additional diagrams of the FVVCS process are given by Fig. 3.5, see the Feynman rules in Appendix B.

The four-Fermi interaction can inspire an interpretation of the UV completion proposed in this work. Assume that causality and unitary should render the LSR exactly. Then, the four-point interaction that completes the FVVCS matrix element corresponds to a short range interaction between the involved fields.

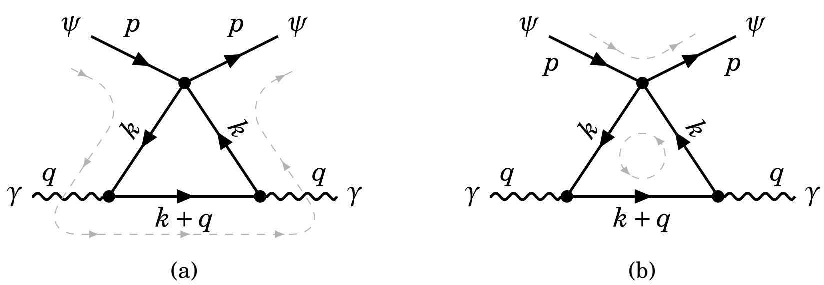


Figure 3.5: **FVVCS Feynman diagrams of the Nambu-Jona-Lasinio interaction**. Incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the momentum of the photon, p is the momentum of the fermion ψ, and k is the unbounded momentum of the one-loop. The dashed lines indicate the charge flow. The crossed diagrams are not displayed.

Chapter 4

Longitudinal Sum Rules for Light-by-Light Scattering

More insights can be gained from the photo-disintegration process of a vector particle if the transversal and longitudinal polarizations of the particle are taken into account. As a result, the attention is focused in this chapter on the Lagrangians

$$\mathscr{L}_4 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + g\bar{\psi}\gamma_\mu\psi B_\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 B_\mu B^\mu,$$

$$\mathscr{L}_8 = \left[(\partial_\mu + \mathrm{i}eA_\mu + \mathrm{i}gB_\mu)\pi\right]^\dagger \left[(\partial^\mu + \mathrm{i}eA^\mu + \mathrm{i}gB^\mu)\pi\right] - m\pi^\dagger\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 B_\mu B^\mu.$$

With the replacement $M^2 \to -\widetilde{Q}^2$, followed by the limit $\widetilde{Q}^2 \to 0$, the vector field B_μ can be viewed as a photon field with $-\widetilde{Q}^2 = p.p < 0$. Thus, the results in this section can be easily translated into results about light-by-light scattering. To allow a consistent transition between the massive vector and massless vector, the redefinition $\nu = \frac{p\cdot q}{M} \to \nu = p\cdot q$ is employed.

4.1 Cross-Sections, Forward Double Virtual Compton Scattering and Dispersion Relations

Consider the vector particle B characterized by the transversal polarizations χ_- and χ_+, and the longitudinal polarization χ_{L}. Regard only the longitudinal polarization of the photon, denoted by ϵ_{L}. Then, for each of the polarizations χ_-, χ_+, and χ_{L} a cross-section can be defined. However, the cross-sections of χ_- and χ_+ are equal. Thus, define $\sigma_{\mathrm{TL}}(\nu, Q^2)$ as the cross-section for the polarizations χ_+ and ϵ_{L}, and $\sigma_{\mathrm{LL}}(\nu, Q^2)$ as the cross-section for the polarizations χ_L and ϵ_{L}. The following relation holds

$$\sigma_{\mathrm{L}}(\nu, Q^2) = \frac{1}{3}\left[2\sigma_{\mathrm{TL}}(\nu, Q^2) + \sigma_{\mathrm{LL}}(\nu, Q^2)\right].$$

For the forward double virtual Compton scattering (FVVCS) process, only the amplitudes

$$T_{\mathrm{TL}}(\nu, Q^2) = \chi_{+\alpha}\chi^*_{+\beta}\epsilon_{\mathrm{L}\mu}\epsilon_{\mathrm{L}\nu}\mathscr{M}^{\mu\nu\alpha\beta}(\nu, Q^2),$$
$$T_{\mathrm{LL}}(\nu, Q^2) = \chi_{\mathrm{L}\alpha}\chi_{\mathrm{L}\beta}\epsilon_{\mathrm{L}\mu}\epsilon_{\mathrm{L}\nu}\mathscr{M}^{\mu\nu\alpha\beta}(\nu, Q^2),$$

can be considered, where $\mathscr{M}^{\mu\nu\alpha\beta}(\nu, Q^2)$ is the general matrix element of FVVCS. The following relation holds

$$T_{\mathrm{L}}(\nu, Q^2) = \frac{1}{3}\left[2T_{\mathrm{TL}}(\nu, Q^2) + T_{\mathrm{LL}}(\nu, Q^2)\right]. \tag{4.1}$$

Causality, unitarity and crossing-symmetry can be exploited, similar to Chapter 3.3, in order to write unsubstracted dispersion relations for $T_{\mathrm{TL}}(\nu, Q^2)$ and $T_{\mathrm{TL}}(\nu, Q^2)$. In the limit $Q^2 \to 0$, these relations have the form

$$\frac{1}{Q^2}T_{\mathrm{LL}}(\nu, Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} - \Delta_{\mathrm{LL}} = \frac{4}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_{\mathrm{LL}}(\nu', Q^2)}{Q^2}\Big|_{Q^2\to 0}, \quad \text{LSR-L} \tag{4.2}$$

$$\frac{1}{Q^2} T_{\rm TL}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} - \Delta_{\rm TL} = \frac{4}{\pi}\int\limits_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_{\rm TL}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0}, \quad \text{LSR-T} \tag{4.3}$$

where

$$\Delta_{\rm LL} = \frac{1}{Q^2} T_{\rm LL}(\nu,Q^2)\Big|_{\substack{\nu\to\infty\\ Q^2\to 0}}, \tag{4.4}$$

$$\Delta_{\rm TL} = \frac{1}{Q^2} T_{\rm TL}(\nu,Q^2)\Big|_{\substack{\nu\to\infty\\ Q^2\to 0}}. \tag{4.5}$$

Note the redefinition $\nu = p\cdot q$. By making an analogy with Chapter 3.3, Eq. 4.2 can be interpreted as the longitudinal sum rule (LSR) form of the longitudinal polarization of a vector particle (LSR-L), and Eq. 4.3 can be interpreted as the LSR form of the transversal polarization of a vector particle (LSR-T).

4.2 Results and Discussions

Due to the increase complexity of the matrix element $\mathscr{M}^{\mu\nu\alpha\beta}(\nu,Q^2)$, the LHS computations of Eqs. 4.2 and 4.3 were carried out using a direct calculation of the tensor one-loop integrals. This approach can be successfully employed in the limit $\nu\to 0$ and $Q^2\to 0$, while its usability in the limit $\nu\to\infty$ is restricted. Hence, $\Delta_{\rm LL}$ and $\Delta_{\rm TL}$ could not be directly calculated[1]. In this section, $\Delta_{\rm LL}$ and $\Delta_{\rm TL}$ are regarded as the difference

$$\Delta_{\rm LL} = \frac{T_{\rm LL}(\nu,Q^2)}{Q^2}\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} - \frac{4}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_{\rm LL}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0},$$

$$\Delta_{\rm TL} = \frac{T_{\rm LL}(\nu,Q^2)}{Q^2}\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} - \frac{4}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_{\rm LL}(\nu',Q^2)}{Q^2}\Big|_{Q^2\to 0}.$$

However, strong clues support Eqs. 4.4 and 4.5, e.g. electro-magnetic gauge invariance and Eq. 4.1. A decomposition of $\mathscr{M}^{\mu\nu\alpha\beta}(\nu,Q^2)$ similar to the algorithm in Appendix F.2 is required to properly compute $\Delta_{\rm LL}$ according to Eq. 4.4, and $\Delta_{\rm TL}$ according to Eq. 4.5.

For the Lagrangians $\mathscr{L}_5$ and $\mathscr{L}_8$, the LSR-L is fulfilled exactly, i.e. $\Delta_{\rm LL} = 0$. In contrast, LSR-T gives $\Delta_{\rm TL} = \frac{3}{2}\Delta_{\rm L}$, with $\Delta_{\rm L}$ in Table 3.2. These result are compatible with the results of the LSR given by Eq. 4.1. Therefore, in the case of a vector particle, $\Delta_{\rm L}$ is generated by the transversal component of $T_{\rm L}(\nu,Q^2)$.

A notable limit is $M^2\to 0$, and it implies that

$$\lim_{M^2\to 0} \frac{1}{Q^2M^2} T_{\rm LL,5}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{g^2e^2}{15\pi^2 m^4} = 96 c_{1,1/2}, \tag{4.6}$$

$$\lim_{M^2\to 0} \frac{1}{Q^2M^2} T_{\rm TL,5}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{g^2e^2}{45\pi^2 m^4} = 32 c_{1,1/2}, \tag{4.7}$$

$$\lim_{M^2\to 0} \frac{1}{Q^2M^2} T_{\rm LL,8}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{7g^2e^2}{760\pi^2 m^4} = 96 c_{1,0}, \tag{4.8}$$

$$\lim_{M^2\to 0} \frac{1}{Q^2M^2} T_{\rm TL,8}(\nu,Q^2)\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = \frac{7g^2e^2}{240\pi^2 m^4} = 32 c_{1,0}. \tag{4.9}$$

The quantities $c_{1,1/2}$ and $c_{1,0}$ are the low energy constants of the Euler-Heisenberg Lagrangian $\mathscr{L}_{\rm EH} = c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2$ for a spin-½ and spin-0 matter particles, respectively[2]. The substitution $M^2 \to -\widetilde{Q}^2$ coverts the above relations into statements about light-by-light scattering.

[1] These calculations were cross-checked by using X-PACKAGE [22]. The capabilities of the X-PACKAGE are also limited when $\nu\to\infty$.

[2] $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta$, where A_μ is the photon field

In this context, Eq. 4.6 to 4.9 are in agreement with the low-energy expansion via an effective Lagrangian discussed in [10], and

$$c_{1,1/2} = \frac{\alpha^2}{90\pi^2 m^4}, \tag{4.10}$$

$$c_{1,0} = \frac{7\alpha^2}{1440\pi^2 m^4}, \tag{4.11}$$

match the known values [24, p. 6-9], where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.

The calculations undertaken for the current work also yielded the constants

$$c_{2,1/2} = \frac{7\alpha^2}{360\pi^2 m^4}, \tag{4.12}$$

$$c_{2,0} = \frac{\alpha^2}{1440\pi^2 m^4}, \tag{4.13}$$

by a similar analysis of the amplitude $\chi_{+\alpha}\chi^*_{-\beta}\epsilon_{-\mu}\epsilon^*_{+\alpha}\mathcal{M}^{\mu\nu\alpha\beta}(\nu,Q^2)$, in the limits $\nu\to 0$, $Q^2\to 0$ and $\tilde{Q}^2\to 0$. The constants $c_{2,1/2}$ and $c_{2,0}$ also match the known values [24, p. 6-9], and the relations obtained for $\chi_{+\alpha}\chi^*_{-\beta}\epsilon_{-\mu}\epsilon^*_{+\alpha}\mathcal{M}^{\mu\nu\alpha\beta}(\nu,Q^2)$ are in compliance with [10].

The above results confirm the low energy expansion of the Euler-Heisenberg Lagrangian in leading order QED. Because the LRS-L holds with $\Delta_{\text{LL}} = 0$, the light-by-light cross-section $\sigma_{\text{LL}}(\nu,Q^2)$ can be related to $c_{1,1/2}$ and $c_{1,0}$ by using Eqs. 4.2, 4.6, 4.8 and by invoking the replacement $M^2\to -\tilde{Q}^2$:

$$c_1 = \frac{1}{24\pi}\int_{\nu_0}^{\infty} d\nu' \, \frac{\sigma_{\text{LL}}(\nu',Q^2)}{Q^2\tilde{Q}^2}\Big|_{\substack{Q^2\to 0\\ \tilde{Q}^2\to 0}}. \tag{4.14}$$

This is not the case for $\sigma_{\text{TL}}(\nu,Q^2)$ because of the additional $\Delta_{\text{TL}}\neq 0$ in LSR-T given by Eq. 4.3. Remark that Eq. 4.14 is a result of leading order QED.

Equations 4.6 to 4.8 point out an interplay between $\frac{T_{\text{TL}}(\nu,Q^2)}{Q^2}\big|_{\substack{\nu\to 0\\ Q^2\to 0}}$ and $\frac{T_{\text{LL}}(\nu,Q^2)}{Q^2}\big|_{\substack{\nu\to 0\\ Q^2\to 0}}$ if $M^2\to 0$

$$\lim_{M^2\to 0}\frac{1}{Q^2M^2}\left[3T_{\text{TL}}(\nu,Q^2) - T_{\text{LL}}(\nu,Q^2)\right]\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = 0. \tag{4.15}$$

For a massless case, the quantity $\Delta_{\text{TL}} = \frac{3}{2}\Delta_{\text{L}}$ can be linked to the cross-sections $\sigma_{\text{LL}}(\nu,Q^2)$ and $\sigma_{\text{TL}}(\nu,Q^2)$ through

$$\lim_{M^2\to 0}\frac{\Delta_{\text{TL}}}{M^2} = \lim_{M^2\to 0}\frac{3\Delta_{\text{L}}}{2M^2} = \lim_{M^2\to 0}\frac{4}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \frac{1}{Q^2M^2}\left[\frac{1}{3}\sigma_{\text{LL}}(\nu',Q^2) - \sigma_{\text{TL}}(\nu',Q^2)\right]\Big|_{Q^2\to 0}. \tag{4.16}$$

The above relations is derived by combining Eqs. 4.15, 4.2 and 4.3, and by assuming that $\Delta_{\text{LL}} = 0$.

In the case of $\mathcal{L}_5$, the computations of $\frac{T_{\text{TL}}(\nu,Q^2)}{Q^2}\big|_{\substack{\nu\to 0\\ Q^2\to 0}}$ and $\frac{T_{\text{LL}}(\nu,Q^2)}{Q^2}\big|_{\substack{\nu\to 0\\ Q^2\to 0}}$ yielded that a form of Eq. 4.15 can be written for any M^2

$$\frac{1}{Q^2}\left[(6m^2 - M^2)T_{\text{TL}}(\nu,Q^2) - 2m^2 T_{\text{LL}}(\nu,Q^2)\right]\Big|_{\substack{\nu\to 0\\ Q^2\to 0}} = 0. \tag{4.17}$$

In consequence, for a massive vector particle, Eq. 4.16 takes the form

$$\Delta_{\text{TL}} = \frac{3}{2}\Delta_{\text{L}} = \frac{4}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \frac{1}{Q^2}\left[\frac{2m^2}{6m^2 - M^2}\sigma_{\text{LL}}(\nu',Q^2) - \sigma_{\text{TL}}(\nu',Q^2)\right]\Big|_{Q^2\to 0}, \tag{4.18}$$

in leading order QED. Note that Eqs. 4.15 and 4.16 are recovered from Eqs. 4.17 and 4.18, respectively, if the limit $M^2\to 0$ is taken.

By employing the change $M^2 \to -\widetilde{Q}^2$, where $\widetilde{Q}$ is the virtuality of the second photon, the following relation can be derived for the light-by light scattering in leading order QED

$$\lim_{\widetilde{Q}^2\to 0} \frac{\Delta_{\text{TL-LbL}}}{\widetilde{Q}^2} = \lim_{\widetilde{Q}^2\to 0} \frac{3\Delta_{\text{L-LbL}}}{2\widetilde{Q}^2} = \lim_{\widetilde{Q}^2\to 0} \frac{4}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{Q^2\widetilde{Q}^2}\left[\frac{1}{3}\sigma_{\text{LL}}(\nu',Q^2) - \sigma_{\text{TL}}(\nu',Q^2)\right]\Big|_{Q^2\to 0}. \tag{4.19}$$

Even though, Eq. 4.19 is satisfied in leading order QED, the hypotheses of the SK theorem and the low energy expansion of the Euler-Heisenberg Lagrangian in [10] suggest a broader validity.

As mentioned before, $\Delta_{\text{TL-LbL}}$ is given by $\frac{T_{\text{TL}}(\nu,Q^2)}{Q^2}|^{\nu\to\infty}_{Q^2\to 0}$, which is a constant, at least in leading order QED. A first question would be if $\Delta_{\text{TL-LbL}} = 0$ when higher order expansions are considered or, even more, when the full, non-perturbative theory is considered. The second question is inspired by the UV completion discussed in Chapter 3.5: Is there a sensible UV completion of the light theory that makes $\Delta_{\text{TL-LbL}} = 0$? Nevertheless, the most stringent questions is if $T_{\text{TL}}(\nu,Q^2)|_{\nu\to\infty}$ can actually depend on Q^2 if the energy of the photon in infinite. If $T_{\text{TL}}(\nu,Q^2)|_{\nu\to\infty} \in \{0,\infty\}$ for any Q^2, then $\Delta_{\text{TL-LbL}} = 0$. Equation 4.19 transforms into a super-convergent sum rule if $\Delta_{\text{TL-LbL}} = \frac{3}{2}\Delta_{\text{L-LbL}} = 0$:

$$0 = \lim_{\substack{Q^2\to 0\\ \widetilde{Q}^2\to 0}} \frac{4}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{Q^2\widetilde{Q}^2}\left[\frac{1}{3}\sigma_{\text{LL}}(\nu',Q^2) - \sigma_{\text{TL}}(\nu',Q^2)\right]. \tag{4.20}$$

The above super-convergent relation is an appealing object of study and raises multiple questions.

Chapter 5

Conclusions and Outlook

Causality and the unitarity of the scattering matrix $\mathcal{S}$ are the fundamental principles that lead to dispersive sum rules. Causality implies that the scattering amplitude must be an analytic function in the complex energy plane, while the unitarity of the scattering matrix connects the imaginary part of the scattering amplitude to the cross-section of the scatterer via the optical theorem. Nevertheless, additional constraints and assumptions on the amplitude of the scattering matrix must be considered when deriving dispersive sum rule. Such a constraint can be the low energy theorem, whereas the high energy behaviour of the scattering amplitude is assumed to be bounded.

Even though, dispersive sum rules should hold for both perturbative and non-perturbative processes, their examination in filed theory is essential due to the additional constraints and assumptions imposed on the scattering amplitude. This examination can confirm the sum rules or it can be used to further refine the constraints or assumptions used in the derivation of the sum rules. In consequence, a first step is to check if the sum rules hold in first order QED.

The current work examines the unsubstracted sum rule for the longitudinal polarization of the photon (LSR) [12; 13]. It considers a large number of models and it investigates the associated photo-disintegration and forward double virtual Compton scattering (FVVCS) in leading order QED. The proposed LSR in [12; 13] is not validated for the majority of the considered models. However, a revised variant of the LSR is proposed, by refining the assumptions on the high energy behaviour of the scattering amplitude. The LSR would hold exactly if the longitudinal amplitude of the scattering matrix does not depend on the virtuality of the photon if the energy of the photon is infinite. In addition, the LSR proposed in this work allows a deeper insight in the light-by-light scattering process and a possible super-convergent sum rule is highlighted in this case.

A new interpretation for the LSR proposed by [12] and [13] is suggested. This interpretation is based on the assumptions of the Sugawara-Kanazawa (SK) theorem: the LSR holds up to the a value Δ_{L} of the amplitude at $\nu \to \infty$. The SK theorem proves to be a suitable setting for the LSR, as it provides a rigorous interpretation of Δ_{L}. In consequence, the LSR is reformulated according to Eq. 3.7. The proposed reformulation of the LSR is compatible with the result obtained in [15].

Equation 3.7 is verified in leading order QED when considering photo-disintegration processes governed by the interaction listed in Table 1.1, see the results in Table 3.2. In Chapter 3.5 examples of renormalizable interactions, e.g. $\mathcal{L}_{7,\text{int}} = g\pi^\dagger\pi\phi$, and non-renormalizibale interactions, e.g. $\mathcal{L}_{3,\text{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi$, that fulfill the LSR with $\Delta_{\text{L}} = 0$ are highlighted. If $\Delta_{\text{L}} \neq 0$, an UV completion of the interaction is proposed by using a four-point type interaction, the lettered entries in Table 3.2. The non-renormalizable Nambu-Jona-Lasinio interaction is suggested as a way to drive the LSR exactly for the example discussed by [15].

The rich structure of the process $\gamma + \text{B} \to \gamma + \text{B}$, where B is a neutral vector particle, enables further insights. Similar to Eq. 3.7, LSR relations can be derived for the amplitudes $T_{\text{LL}}(\nu, Q^2)$ and $T_{\text{TL}}(\nu, Q^2)$, LSR-L given by Eq. 4.2 and LSR-T given by Eq. 4.3, respectively. In leading order QED, it is proven that LSR-L holds exactly, while LSR-T gives rise to the quantity $\Delta_{\text{TL}} = \frac{3}{2}\Delta_{\text{L}}$.

An interesting limit discussed in Chapter 4.2 is $M^2 \to 0$ for $\gamma + \mathrm{B} \to \mathrm{P} + \bar{\mathrm{P}} \to \gamma + \mathrm{B}$. In this case, a recast $M^2 \to -\tilde{Q}^2$ grants access to the light-by-light scattering. The amplitudes $T_{\mathrm{LL}}(\nu, Q^2)$ and $T_{\mathrm{TL}}(\nu, Q^2)$ of the light-by-light scattering enforce in leading order QED the low energy expansion of the Euler-Heisenberg Lagrangian examined in [10]. Through $T_{\mathrm{LL}}(\nu, Q^2)$ and $T_{\mathrm{TL}}(\nu, Q^2)$, the constant c_1 of the Euler-Heisenberg Lagrangian is recovered to the known values of QED or scalar QED.

In the light-by-light scattering, Eq. 4.15 combined with LSR-L and LSR-T yield a relation which links $\Delta_{\mathrm{TL}} = \frac{3}{2}\Delta_{\mathrm{L}}$ to the cross-sections $\sigma_{\mathrm{LL}}(\nu, Q^2)$ and $\sigma_{\mathrm{TL}}(\nu, Q^2)$, Eqs. 4.16 and 4.19. If $\Delta_{\mathrm{L}} = 0$, then Eq. 4.19 would transforms into a super-convergent sum rule given by Eq. 4.20. Even though Eq. 4.16 is a result of leading order QED, it could prove of theoretical value due to the more general statements of the low energy expansion in [10] and SK theorem.

Further developments of the new form proposed for LSR Eq. 3.7, with the LSR-L and LSR-T variations, could include higher order calculations of the photo-disintegration process and of the FVVCS process. The photo-disintegration of charged particles can add useful observations to the LSR. The Compton scattering $\gamma + \mathrm{P} \to \gamma + \mathrm{P}$, where P is a charged particle, should also be investigated for interactions beyond that analyzed in [15]. This investigation might bring to light other interactions for which the LSR holds exactly and might reconfirm the validity of Eq. 3.7.

The LSR deserves also an extension to the entire Standard Model framework, especially for the photo-disintegration of the Z-boson, or to non-Abelian gauge theories. Would the Higgs particle ensure a null Δ_{L}? In addition, Table 3.2 points out that a super-symmetric theory might have all the ingredients for an exact LSR.

Appendix A

Tree Diagram and FVVCS Diagrams Used by Llanta and Tarrach

Figure A.1 highlights the Feynman diagrams used by Llanta and Tarrach in [15]. These diagrams are not explicitly present in their paper, yet they can be inferred from the process the authors analyse and the statements the authors make in their article.

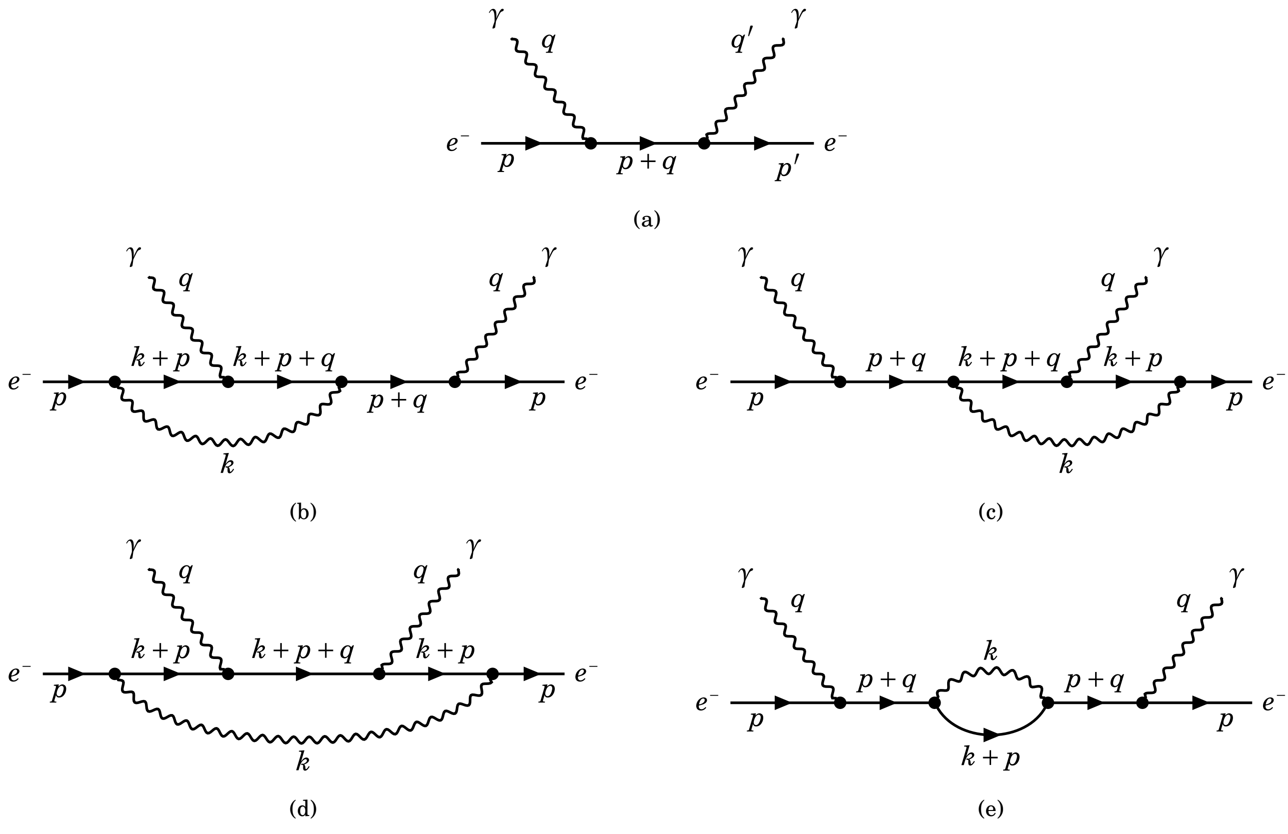


Figure A.1: **Tree and FVVCS Feynman diagrams of the interaction analysed in [15].** (a) Tree diagram. q and p are the momenta of the incoming photon and electron, respectively. q' and p' are the momenta of the outgoing photon and electron, respectively. (b), (c), (d), and (e) FVVCS diagrams. Incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the momentum of the photon, p is the momentum of the electron, and k is the unbounded momentum of the one-loop. The crossed-diagrams are not shown.

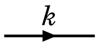
Appendix B

Feynman Rules

This appendix lists the Feynman rules corresponding to the interaction Lagrangians used throughout the current paper. The coupling g of the interaction Lagrangian is assumed to be constant. The fields are label as: ψ – fermion field, π – charged scalar particle field, V_μ – vector particle field, ϕ – scalar particle field.

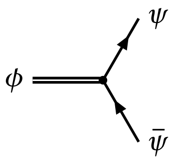
B.1 Interactions Involving Fermions: $\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi + \mathscr{L}_{\mathrm{int}}$

A. Propagators

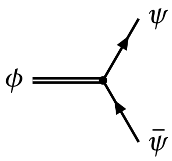
1. Fermion propagator: 

$= \mathrm{i}\dfrac{\not{k} + m}{k^2 - m^2 + \mathrm{i}0^+}.$

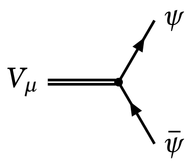
B. Interaction Lagrangians

1. $\mathscr{L}_{\mathrm{int}} = g\bar{\psi}\psi\phi$: 

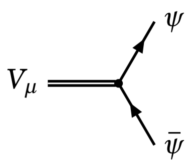
$= \mathrm{i}g,$

2. $\mathscr{L}_{\mathrm{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi$: 

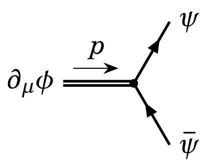
$= -g\gamma_5,$

3. $\mathscr{L}_{\mathrm{int}} = g\bar{\psi}\gamma^\mu\psi V_\mu$: 

$= \mathrm{i}g\gamma^\mu,$

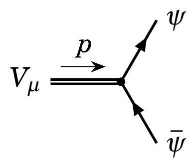
4. $\mathscr{L}_{\mathrm{int}} = g\bar{\psi}\gamma^\mu\gamma_5\psi V_\mu$: 

$= \mathrm{i}g\gamma^\mu\gamma_5,$

5. $\mathscr{L}_{\mathrm{int}} = g\bar{\psi}\gamma^\mu\gamma_5\psi\partial_\mu\phi$: 

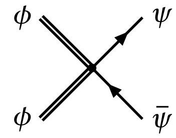
$= gp_\mu\gamma^\mu\gamma_5,$

6. $\mathscr{L}_{\text{int}} = g\bar{\psi}\sigma^{\mu\nu}\psi(\partial_\mu V_\nu - \partial_\nu V_\mu)$:



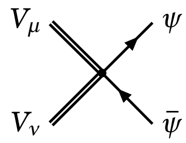
$= 2\mathrm{i}g p_\nu \sigma^{\mu\nu}$,

7. $\mathscr{L}_{\text{int}} = g\bar{\psi}\psi\phi^2$:



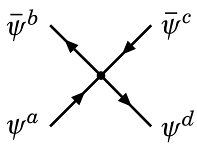
$= 2\mathrm{i}g$,

8. $\mathscr{L}_{\text{int}} = g\bar{\psi}\psi V_\mu V^\mu$:



$= 2g g_{\mu\nu}\gamma_5$,

9. $\mathscr{L}_{\text{int}} = g\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right]$:

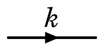


$= 2g\mathrm{i}\Big[(\delta^{ad}\delta^{bc} - \delta^{ab}\delta^{cd}) - (\gamma_5^{ad}\gamma_5^{bc} - \gamma_5^{ab}\gamma_5^{cd})\Big]$ [25].

B.2 Interactions Involving Charged Scalars: $\mathscr{L} = \partial_\mu\pi^\dagger\partial^\mu\pi - m^2\pi^\dagger\pi + \mathscr{L}_{\text{int}}$

A. Propagators

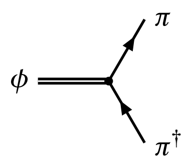
1. Charged scalar propagator:



$= \mathrm{i}\dfrac{1}{k^2 - m^2 + \mathrm{i}\,0^+}$.

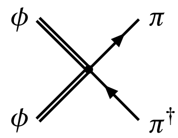
B. Interaction Lagrangians

1. $\mathscr{L}_{\text{int}} = g\pi^\dagger\pi\phi$:



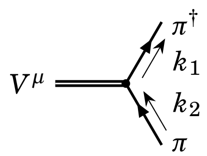
$= \mathrm{i}g$,

2. $\mathscr{L}_{\text{int}} = g\pi^\dagger\pi\phi^2$:



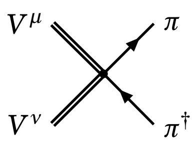
$= 2\mathrm{i}g$,

3. $\mathscr{L}_{\text{int}} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)V^\mu$:



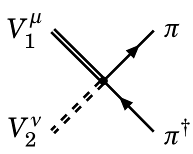
$= -\mathrm{i}g(k_1^\mu + k_2^\mu)$,

4. $\mathscr{L}_{\text{int}} = g\pi^\dagger\pi V_\mu V^\mu$:



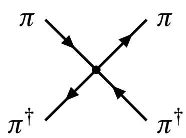
$= 2\mathrm{i}g^2 g^{\mu\nu}$,

5. $\mathscr{L}_{\text{int}} = g_1 g_2\pi^\dagger\pi V_{1\mu}V_2^\mu$:



$= \mathrm{i}g_1 g_2 g^{\mu\nu}$,

6. $\mathscr{L}_{\text{int}} = \frac{1}{4}g(\pi^\dagger\pi)^2$:



$= \mathrm{i}g$.

Appendix C

Kinematics and Cross-Sections

C.1 Kinematics of $1+2 \to 3+4$ Processes

The kinematic process of inters for this paper is the scattering of two incoming particles resulting in two outgoing particle: $1+2 \to 3+4$ process. When the outgoing particles are identical to the incoming ones, the process is called elastic scattering. When at least one outgoing particle is different from the incoming particles, the process is called inelastic scattering.

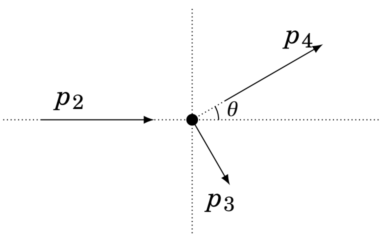
To describe this process realistically, the most commonly used reference systems are the laboratory reference system (LRS) and the center-of-momentum reference system (COM-RS or simply COM), both depicted in Fig. C.1. The calculations for this paper were done mostly in COM, see Appendix D.1.1. COM was chosen because the outgoing particles have the same mass. In this reference frame, if the process has a 2π rotational symmetry around the axis of the incoming particles, the momenta of the particles read

$$p_1 = \begin{bmatrix} E_1 \\ 0 \\ 0 \\ -|\mathbf{p}_1| \end{bmatrix}, \quad p_2 = \begin{bmatrix} E_2 \\ 0 \\ 0 \\ -|\mathbf{p}_2| \end{bmatrix}, \quad p_3 = \begin{bmatrix} E_3 \\ -|\mathbf{p}_3|\sin(\theta) \\ 0 \\ -|\mathbf{p}_3|\cos(\theta) \end{bmatrix}, \quad p_4 = \begin{bmatrix} E_4 \\ |\mathbf{p}_4|\sin(\theta) \\ 0 \\ |\mathbf{p}_4|\cos(\theta) \end{bmatrix}, \tag{C.1}$$

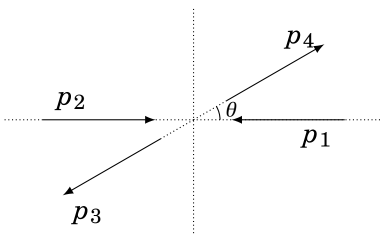
with $|\mathbf{p}_1| = |\mathbf{p}_1|$ and $|\mathbf{p}_3| = |\mathbf{p}_4|$. In the above, the z-axis was considered as the axis of the of the incoming particles.

A suitable reference system and the use of symmetries imply simpler computations. On the other hand, it is desirable to express the result in a from that is independent of the reference system. This aim can be achieved by recasting the reference frame-dependent quantities into reference frame-independent ones.

In the case of a $1+2 \to 3+4$ process, the Mandelstam variables are useful reference frame-



(a) Laboratory reference system



(b) Center-of-momentum reference system

Figure C.1: **Kinematics of a $1+2 \to 3+4$ process in LRS and COM.** In LRS, Fig. C.1a, one of the initial particle is at rest: $p_1 = [m_1, \mathbf{0}]$ with m_1 the particle mass. In COM, Fig. C.1b, the spatial momenta of the initial and final particles, respectively, add to null: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$ and $\mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$.

independent quantities:

$$\begin{aligned} s &\equiv (p_1+p_2)^2 = (p_3+p_4)^2, \\ t &\equiv (p_1-p_3)^2 = (p_2-p_4)^2, \\ u &\equiv (p_1-p_4)^2 = (p_2-p_3)^2, \end{aligned}$$

and satisfy the relation

$$s+t+u = p_1^2+p_2^2+p_3^2+p_4^2 = m_1^2+m_2^2+m_3^2+m_4^2.$$

C.2 Cross-Sections

The cross-section is intrinsic to the scattering of two particles because it can characterize the process uniquely, and because it is independent of the experimental setup. Thus, the cross-section is fundamental to QFT, as it allows to study the interaction, and the interaction products, between two particles.

With a change in the reference system, one of the particles can be considered stationary, denoted target, an the other mobile, denoted incoming particle. The incoming particle is usually part of a batch of beam particles with a given flux. With this notions, the classical definition of the cross-section [26, p. 70]

$$\sigma = \frac{\text{Number of Interactions per Unit of Time per Target}}{\text{Incident Flux}}$$

can be adjusted in the form of a differential formula and extended to QFT by

$$\mathrm{d}\sigma = \frac{\mathrm{d}P}{T\,\Phi} = \frac{\text{Probability Density of the Interaction per Unit of Time}}{\text{Incident Flux}}. \tag{C.2}$$

The extension considers that the interaction takes place always between two particles that both occupy a large volume V, for a long period of time T. These considerations imply that the beam has only one particle, thus the flux Φ has to be normalized as such [18, p. 58].

Consider the scattering process: $2-\text{initial particles} \to n-\text{final particles} \equiv p_{\mathrm{i}1}+p_{\mathrm{i}2} \to p_{\mathrm{f}1}+p_{\mathrm{f}2}+p_{\mathrm{f}3}+\cdots$. Using Eq. C.2, the differential cross-section of such a process takes the form [18, p. 61]

$$\mathrm{d}\sigma = \frac{1}{(2E_{\mathrm{i}1})(2E_{\mathrm{i}2})|\mathbf{v}_{\mathrm{i}1}-\mathbf{v}_{\mathrm{i}2}|}|\mathcal{M}|^2\,\mathrm{d}\Pi_{\mathrm{LIPS}}, \tag{C.3}$$

where (i) $E_{\mathrm{i}1}$ and $E_{\mathrm{i}2}$ are the energies of the particles $p_{\mathrm{i}1}$ and $p_{\mathrm{i}2}$, respectively. (ii) $|\mathbf{v}_{\mathrm{i}1}-\mathbf{v}_{\mathrm{i}2}|$ is the relative velocity between the incoming particle and the target. (iii) $\mathcal{M}$ is the matrix element of the transition $\langle p_{\mathrm{f}1}p_{\mathrm{f}2}p_{\mathrm{f}3}\cdots|p_{\mathrm{i}1}p_{\mathrm{i}2}\rangle$. (iv) $\Pi_{\mathrm{LIPS}} \equiv \prod_{\mathrm{f}} \frac{\mathrm{d}^3 p_{\mathrm{f}}}{(2\pi)^3}\frac{1}{2E_{\mathrm{f}}}(2\pi)^4\delta^4(p_{\mathrm{i}1}+p_{\mathrm{i}2}-\sum_{\mathrm{f}} p_f)$ is the Lorenz-invarinat phase space (LISP).

To determine the total cross-section of the two-body scattering, preform the integration of Eq. C.3 over LIPS

$$\sigma = \frac{1}{(2E_{\mathrm{i}1})(2E_{\mathrm{i}2})|\mathbf{v}_{\mathrm{i}1}-\mathbf{v}_{\mathrm{i}2}|}\int \mathrm{d}\Pi_{\mathrm{LIPS}}|\mathcal{M}|^2. \tag{C.4}$$

$\mathcal{M}$, $\int \mathrm{d}\Pi_{\mathrm{LIPS}}$, and $4E_{\mathrm{i}1}E_{\mathrm{i}2}|\mathbf{v}_{\mathrm{i}1}-\mathbf{v}_{\mathrm{i}2}| = 4\left[(p_{\mathrm{i}1}\cdot p_{\mathrm{i}2})^2 - m_{\mathrm{i}1}^2 m_{\mathrm{i}2}^2\right]^{1/2}$ are Lorenz-invariant, therefore the total cross-section σ of Eq. C.4 is also Lorenz-invariant.

Lorenz-invariance permits simpler computations for σ by choosing a suitable reference frame. Nevertheless, the LIPS integration is not a trivial task, but for a $1+2\to 3+4$ scattering the integration is manageable. The total cross-section σ in COM, the reference-frame chosen for computations in this paper, reads as [26, p. 72]

$$\sigma = \frac{1}{64\pi^2 s}\frac{\mathbf{p}_{\mathrm{f}3}}{\mathbf{p}_{\mathrm{i}1}}\int \mathrm{d}\Omega|\mathcal{M}|^2, \tag{C.5}$$

where $s = (p_{\text{i1}} + p_{\text{i2}})^2$, and Ω is the solid angle corresponding to the scattering of particle p_{f3}.If scattering process has a rotational symmetry around the incoming axis of the initial particles, Eq. C.5 can be simplified to

$$\sigma = \frac{1}{32\pi s} \frac{\mathbf{p}_{\text{f3}}}{\mathbf{p}_{\text{i1}}} \int_0^\pi \mathrm{d}\theta \cos(\theta) |\mathcal{M}|^2, \tag{C.6}$$

with θ being the angle between the direction of the final particle p_{f3} and the direction of the initial particle p_{i1}.

Appendix D

Photo-Disintegration: Tree-Level Cross Sections and Forward Double Virtual Compton Scattering

Photo-disintegration is the process through which a composite particle breaks up under the influence of a light quanta, i.e. $\gamma + \mathrm{P} \to \mathrm{a} + \mathrm{b} + \cdots$. In this work, photo-disintegration represents the process $\gamma + \mathrm{P} \to \mathrm{X}$, where X is a fermion pair or a charged scalar particle pair.

D.1 Tree-Level Cross-Sections of the Photo-Disintegration Process

In the following, the longitudinal and transversal cross-sections (σ_{L} and σ_{T}, respectively) of the photo-disintegration process at tree level are determined. The cross-sections σ_{L} and σ_{T} serve in the analysis carried in Chapter 3.

D.1.1 Cross-Sections Using Tree Level Feynman Diagrams

This section presents only the calculations for the $\sigma(\nu) = \sigma_{\mathrm{T}}(\nu, Q^2 = 0)$ photo-disintegration with an on-shell photon governed by the Lagrangian

$$\mathscr{L}_2 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + \mathrm{i}g\bar{\psi}\gamma_5\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2.$$

Using tree diagrams in Fig. D.1, the matrix element for the two cases is given by

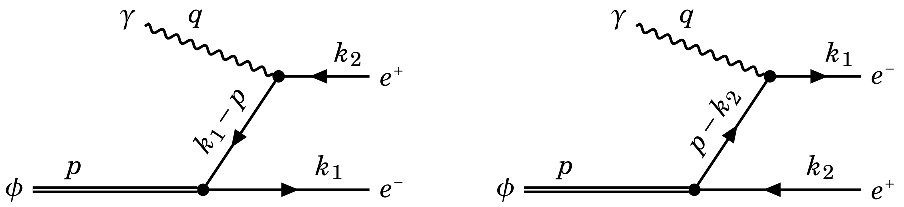
$$\mathrm{i}\mathscr{M} = -ge\epsilon_\mu\bar{u}_1\left[\frac{\gamma_5(\not{k}_1 - \not{p} + m)\gamma^\mu}{(k_1 - p)^2 - m^2} + \frac{\gamma^\mu(\not{p} - \not{k}_2 + m)\gamma_5}{(p - k_2)^2 - m^2}\right]v_2.$$

The matrix element squared and summed over spins and polarization has the form

$$\begin{aligned}|\mathscr{M}|^2 &= \frac{1}{2}g^2e^2\,\mathrm{Tr}\left\{(\not{p}_1 + m)\left[\frac{\gamma_5(\not{k}_1 - \not{p} + m)\gamma^\mu}{(k_1 - p)^2 - m^2} + \frac{\gamma^\mu(\not{p} - \not{k}_2 + m)\gamma_5}{(p - k_2)^2 - m^2}\right](\not{p}_2 - m)\begin{pmatrix}\gamma^\mu \leftrightarrow \gamma_5\\ \gamma^\mu \to \gamma_\mu\end{pmatrix}\right\}\\ &= g^2e^2\left[-\frac{s^2 + M^4}{(t - m^2)(u - m^2)} + \frac{2m^2M^2(s - M^2)^2}{(t - m^2)^2(u - m^2)^2}\right].\end{aligned}$$

where s, t, u are the Mandelstam variables. $|\mathscr{M}|^2$ is then introduced in Eq. C.6 and the integration is preformed. Thus, the cross-section reads

$$\sigma = \frac{g^2e^2}{2\pi(s - M^2)^3}\left[-sM^2\sqrt{1 - \frac{4m^2}{s}} + (s^2 + M^4 - 4m^2M^2)\,\mathrm{arcosh}\left(\frac{\sqrt{s}}{2m}\right)\right]. \tag{D.1}$$



(a) Photon interacts with antifermion (b) Photon interacts with antifermion

Figure D.1: **Tree level diagram of the photo-disintegration process.** The four-momenta of the particles are: q for photon γ, p for the scalar ϕ, k_1 for fermion, and k_2 for anti-fermion.

D.1.2 On-Shell Validation of the Cross-Sections via Optical Theorem

The on-shell σ_{T} for the photo-disintegration process of $\mathscr{L}_2$, Eq. D.1, can be cross-checked by employing the optical theorem, see Chapter 2.1. The optical theorem relates the imaginary part of the forward elastic scattering amplitude of the diagrams in Fig. D.2 to the total cross-section of the tree level diagrams in Fig. D.1:

$$\sigma = \frac{\mathrm{Im}\,\mathscr{M}(s,t=0)}{2E_{\mathrm{CM}}\,|\mathbf{P}_{\mathrm{CM}}|}. \tag{D.2}$$

The above equation is a reformulation of Eq. 2.9 in COM frame. The matrix element of the diagrams in Fig. D.2a, and Fig. D.2b are given by

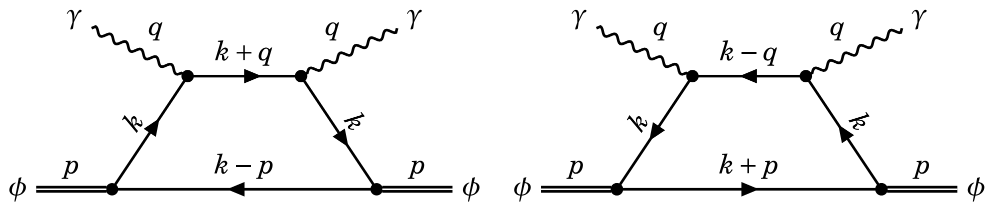
$$\mathrm{i}\mathscr{M} = \mathrm{i}\mathscr{M}_{\mathrm{a}} + \mathrm{i}\mathscr{M}_{\mathrm{b}}, \tag{D.3}$$

with

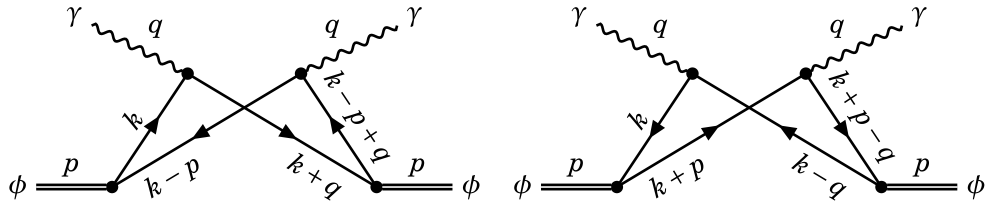
$$\begin{aligned}
\mathrm{i}\mathscr{M}_{\mathrm{a}} = -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}\Bigg\{&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^5(\not{k}-\not{p}+m)\gamma^5(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]^2}\\
&+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}+m)\gamma^5(\not{k}+\not{p}+m)\gamma^5(\not{k}+m)\gamma^\mu(\not{k}-\not{q}+m)\right]}{\left[(k+p)^2-m^2\right]\left[(k-q)^2-m^2\right]\left[k^2-m^2\right]^2}\Bigg\} && \text{(Step 1)}\\
= -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}\Bigg\{&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^5(\not{k}-\not{p}+m)\gamma^5(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]^2}\\
&+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}-m)\gamma^5(\not{k}-\not{p}-m)\gamma^5(\not{k}-m)\gamma^\mu(\not{k}+\not{q}-m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]^2}\Bigg\} && \text{(Step 2)}\\
= -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}&\mathscr{M}_{\mathrm{a}}^{\mu\nu},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{i}\mathscr{M}_{\mathrm{b}} = -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}\Bigg\{&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^5(\not{k}-\not{p}+m)\gamma^\nu(\not{k}-\not{p}+\not{q}+m)\gamma^5(\not{k}+\not{q}+m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]\left[(k-p+q)^2-m^2\right]}\\
&+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}+\not{p}+m)\gamma^5(\not{k}+m)\gamma^\mu(\not{k}-\not{q}+m)\gamma^5(\not{k}-\not{q}+\not{p}+m)\right]}{\left[(k+p)^2-m^2\right]\left[(k-q)^2-m^2\right]\left[k^2-m^2\right]\left[(k+p-q)^2-m^2\right]}\Bigg\} && \text{(Step 1)}\\
= -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}\Bigg\{&\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^5(\not{k}-\not{p}+m)\gamma^\nu(\not{k}-\not{p}+\not{q}+m)\gamma^5(\not{k}+\not{q}+m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]\left[(k-p+q)^2-m^2\right]}\\
&+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}+\not{p}-m)\gamma^5(\not{k}-m)\gamma^\mu(\not{k}-\not{q}-m)\gamma^5(\not{k}-\not{q}+\not{p}-m)\right]}{\left[(k-p)^2-m^2\right]\left[(k+q)^2-m^2\right]\left[k^2-m^2\right]\left[(k-p+q)^2-m^2\right]}\Bigg\} && \text{(Step 2)}\\
= -g^2e^2\epsilon_{1\mu}\epsilon^*_{2\nu}&\mathscr{M}_{\mathrm{b}}^{\mu\nu}.
\end{aligned}$$



(a) Bag diagram



(b) Cat ears diagram

Figure D.2: **One-loop diagram of the FVVCS process**. q, and p represent the four-momenta of the photon γ, and scalar ϕ, respectively. k is the unbounded four-momentum of the one-loop.

In Step 1 of the last two equations the change of variable $k \to -k$ is preformed. Step 2 uses the fact that the traces in the integrals are equal.

$\mathscr{M} = \epsilon_{1\mu}\epsilon^*_{2\nu}\mathscr{M}^{\mu\nu} = \epsilon_{1\mu}\epsilon^*_{2\nu}(\mathscr{M}^{\mu\nu}_{\text{a}} + \mathscr{M}^{\mu\nu}_{\text{b}})$ can be reduced to a sum of scalar integrals. Firstly, note that the circular polarizations $\epsilon_{1\mu}$ and $\epsilon^*_{2\nu}$ of the photons can be chosen such that $\epsilon_{1\mu}q^\mu = \epsilon^*_{2\nu}q^\nu = 0$. In the LRS or COM frame, the scalar product of these polarizations with the four-momentum of the particle ϕ is also zero, i.e. $\epsilon_{1\mu}p^\mu = \epsilon^*_{2\nu}p^\nu = 0$. Secondly, $\mathscr{M}^{\mu\nu}$ can be written in its most general form as

$$\mathscr{M}^{\mu\nu} = p^\mu p^\nu I_1 + q^\mu q^\nu I_2 + p^\mu q^\nu I_3 + q^\mu p^\nu I_4 + g^{\mu\nu} I_5, \tag{D.4}$$

due to its tensorial character. Combining the previous two observations, after the contraction $\epsilon_{1\mu}\epsilon^*_{2\nu}\mathscr{M}^{\mu\nu}$ only the term I_5 remains, thus is enough to determine I_5.

I_5 can be computed from the set of linear equations yielded by contracting $\mathscr{M}^{\mu\nu}$ with $q^\mu p^\nu$,$q^\mu p^\nu$, $p^\mu q^\nu$, and $g^{\mu\nu}$, respectively. Using $q^2 = 0$, the set of linear equation is

$$\begin{cases} q_\mu q_\nu \mathscr{M}^{\mu\nu} = (p\cdot q)^2 I_1, \\ p_\mu q_\nu \mathscr{M}^{\mu\nu} = p^2 p\cdot q I_1 + (p\cdot q)^2 I_4 + p\cdot q I_5, \\ q_\mu p_\nu \mathscr{M}^{\mu\nu} = p^2 p\cdot q I_1 + (p\cdot q)^2 I_3 + p\cdot q I_5, \\ \quad g_{\mu\nu}\mathscr{M}^{\mu\nu} = p^2 I_1 + (p\cdot q) I_3 + (p\cdot q) I_4 + 4 I_5. \end{cases} \tag{D.5}$$

After solving for I_5 in the above equation, the result is:

$$I_5 = \frac{1}{2(p\cdot q)^2}\left[p^2 q_\mu q_\nu - p\cdot q\left(p_\mu q_\nu + q_\mu p_\nu\right) + (p\cdot q)^2 g_{\mu\nu}\right]\mathscr{M}^{\mu\nu}. \tag{D.6}$$

$$\begin{aligned} D_1 &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\left[k^2 - m^2\right]\left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]}, \\ D_2 &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\left[(k+q)^2 - m^2\right]\left[(k-q)^2 - m^2\right]\left[(k+q-p)^2 - m^2\right]}, \\ D_3 &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\left[k^2 - m^2\right]\left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]^2}. \end{aligned} \tag{D.7}$$

At this point the algorithm in Appendix F.2 can be employed. Only the imaginary part of Eq. D.3 has to be computed, thus for the kinematic region $s > 4m^2$ and $M < 2m$ only the integrals

D_1, D_2, and D_3, given in Eq. D.7, have an imaginary part. With the on-shell relations for the forward elastic scattering, i.e $p\cdot p = M^2$, $q\cdot q = 0$, $2p\cdot q = s - M^2$, the imaginary part of $\mathscr{M}$ reads as

$$\begin{aligned}\mathrm{Im}\,\mathscr{M} = 2g^2e^2\epsilon_1\cdot\epsilon_2^*\Bigg\{&-\frac{4\left(-2m^2M^2+M^4+s^2\right)}{s-M^2}\,\mathrm{Im}\,D_1\\ &+\frac{8m^2M^2}{s-M^2}\,\mathrm{Im}\,D_2 - 8m^2M^2\,\mathrm{Im}\,D_3\Bigg\},\end{aligned} \tag{D.8}$$

with

$$\mathrm{Im}\,D_1 = \mathrm{Im}\left[-\frac{1}{32\pi^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1}{m^2+yx\left(s-M^2\right)-sx+sx^2}\right] = -\frac{1}{16\pi}\,\frac{2\,\mathrm{arcosh}\left(\frac{\sqrt{s}}{2m}\right)}{s-M^2},$$

$$\mathrm{Im}\,D_2 = \mathrm{Im}\left[-\frac{1}{32\pi^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1}{m^2+yx\left(s-M^2\right)-sx+sx^2}\right] = -\frac{1}{16\pi}\,\frac{2\,\mathrm{arcosh}\left(\frac{\sqrt{s}}{2m}\right)}{s-M^2},$$

$$\mathrm{Im}\,D_3 = \mathrm{Im}\left[\frac{1}{16\pi^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{y}{\left(m^2+yx\left(s-M^2\right)-sx+sx^2\right)^2}\right] = \frac{1}{32\pi}\,\frac{s\sqrt{1-\frac{4m^2}{s}}}{m^2\left(s-M^2\right)^2}.$$

By replacing the above relations in Eq. D.2 and summing over the photon polarizations, the transversal cross-section is

$$\sigma = \frac{g^2e^2}{2\pi\left(s-M^2\right)^3}\left[-M^2s\sqrt{1-\frac{4m^2}{s}}+\left(s^2+M^4-4m^2M^2\right)\mathrm{arcosh}\left(\frac{\sqrt{s}}{2m}\right)\right]. \tag{D.9}$$

The above cross-section is identical with Eq. D.1.

D.1.3 A Plethora of Cross-Sections

For all the photo-disintegration processes discussed in this work, the cross-section computations can be done in the same straightforward way. The tree diagrams are similar to the ones in Fig. D.1, with an additional seagull diagram in Fig. D.3 for the $\gamma + B \to \pi^\dagger + \pi$ process. In the COM, the following photon polarizations can be chosen [27, p. 185]

$$\underbrace{\epsilon_+ = -\frac{1}{\sqrt{2}}\begin{bmatrix}0\\+1\\+\mathrm{i}\\0\end{bmatrix},\ \epsilon_- = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\+1\\-\mathrm{i}\\0\end{bmatrix}}_{\substack{\text{circular}\\ \text{transversal polarizations}}},\qquad \underbrace{\epsilon_\mathrm{L} = -\frac{1}{\sqrt{-q^2}}\begin{bmatrix}\sqrt{\omega^2-q^2}\\0\\0\\\omega\end{bmatrix} = -\frac{1}{\sqrt{Q^2}}\begin{bmatrix}\sqrt{\omega^2+Q^2}\\0\\0\\\omega\end{bmatrix}}_{\text{longitudinal polarization}},$$

where q is the four-momentum of the photon and $Q^2 = -q^2 > 0$, and ω is the photon energy in COM.

Note hat a massive vector particle B of mass M has three polarizations χ_i^μ. In the given case, they can be chosen in the COM frame as

$$\underbrace{\chi_+ = -\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\-\mathrm{i}\\0\end{bmatrix},\ \chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\+\mathrm{i}\\0\end{bmatrix}}_{\substack{\text{circular}\\ \text{transversal polarizations}}},\qquad \underbrace{\chi_\mathrm{L} = \frac{1}{M}\begin{bmatrix}-\sqrt{\omega^2+Q^2}\\0\\0\\\sqrt{M^2+\omega^2+Q^2}\end{bmatrix}}_{\text{longitudinal polarization}}.$$

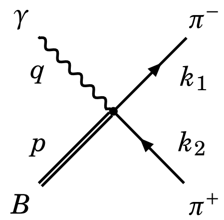


Figure D.3: **Seagull diagram for the photo-disintegration process $\gamma + B \to \pi^\dagger + \pi$.** The four-momenta of the particles are: q for the photon, p for the massive vector, k_1 for the charged scalar particle, and k_2 for the charged scalar anti-particle.

The matrix element averaged over the massive vector polarizations can be constructed by contracting the matrix element with

$$\sum_{i=1,2,3} \chi_i^{*\mu} \chi_i^{\nu} = \frac{1}{3}\left(-g^{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right), \tag{D.10}$$

where p is the four momentum of the particle.

The list below contains the cross-sections $\sigma_{\rm T}$ and $\sigma_{\rm L}$ for any value of Q only for $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_4$, and $\mathscr{L}_5$. Table D.1 lists $\sigma_{\rm T}|_{Q^2\to 0}$ and $\frac{\sigma_{\rm L}}{Q^2}|_{Q^2\to 0}$ for all the interactions considered in this paper. Table D.2 lists $\frac{\sigma_{\rm LL}}{Q^2}|_{Q^2\to 0}$ and $\frac{\sigma_{\rm TL}}{Q^2}|_{Q^2\to 0}$ for the photo-disintegration of a vector particle.

For convenience, the following notations are introduced:

$$\begin{aligned}
s &= (p+q)^2 = M^2 - Q^2 + 2M\nu, \\
v &= \sqrt{1 - \frac{4m^2}{s}}, \\
\beta &= \frac{s - M^2 - Q^2}{2s}, \\
\lambda &= \frac{\sqrt{M^4 + 2M^2\left(Q^2 - s\right) + \left(Q^2 + s\right)^2}}{2s}, \\
\kappa &= \frac{\lambda^2 - \beta^2 + \beta}{\lambda v}.
\end{aligned}$$

1. $\mathscr{L}_{1,\rm int} = g\bar{\psi}\psi\phi$:

$$\begin{aligned}
\sigma_{\rm L} = e^2 g^2 \frac{\beta^2 - \lambda^2}{8\pi\lambda^4 s} \Bigg\{ &- \frac{\left((1-\beta)^2 - \lambda^2\right)\left(v^2 - (2-\beta)\beta - \lambda^2\right)}{\left(\kappa^2 - 1\right)\lambda v} \\
&+ \frac{\lambda^2 + (2-\beta)\beta + v^2 - 3 + (1-\beta)^2\left(v^2 - (2-\beta)\beta\right)}{\left(\kappa^2 - 1\right)v^2} \operatorname{arccoth}(\kappa) \Bigg\}
\end{aligned} \tag{D.11}$$

$$\begin{aligned}
\sigma_{\rm T} = \frac{e^2 g^2}{8\pi\lambda^5 v s} \Bigg\{ &\frac{\left(v^2 - (2-\beta)\beta - \lambda^2\right)\left(\lambda^4 + \lambda^2\left(v^2 - 2\beta\right) - (1-\beta)^2\beta^2\right)}{\left(\kappa^2 - 1\right)} \\
&+ \frac{1}{\kappa}\left[\lambda^6 + \lambda^4\left(2v^2 - \beta^2\right) - \lambda^2\left((\beta - 4)\beta^3 + \beta^2 + 3\beta^2 v^2 - v^4\right)\right. \\
&\left. + (1-\beta)^2\beta^2\left((\beta - 2)\beta + v^2\right)\right] \operatorname{arccoth}(\kappa) \Bigg\}
\end{aligned} \tag{D.12}$$

2. $\mathscr{L}_{2,\rm int} = {\rm i} g\bar{\psi}\psi\gamma_5\phi$:

$$\sigma_{\rm L} = e^2 g^2 \frac{\left(\beta^2 - \lambda^2\right)\left(1 - \beta^2 + \lambda^2\right)^2}{8\pi\beta\lambda^4 s v} \left\{ -\frac{1}{\kappa^2 - 1} + \frac{1}{\kappa} \operatorname{arccoth}(\kappa) \right\} \tag{D.13}$$

$$
\begin{aligned}
\sigma_{\mathrm{T}} = \frac{e^2 g^2}{8\pi\lambda^5 v s} \Bigg\{ & \frac{\left(\lambda^2-(1-\beta)^2\right)\left(\lambda^4+\lambda^2\left(v^2-2\beta\right)-(1-\beta)^2\beta^2\right)}{\left(\kappa^2-1\right)} \\
& -\frac{1}{\kappa}\left[\lambda^6-\lambda^4\left(\beta^2+v^2-3\right)+\lambda^2(1-\beta)^2\left(v^2+(2-\beta)\beta\right)\right. \\
& \left.+(1-\beta)^4\beta^2\right]\operatorname{arccoth}(\kappa)\Bigg\}
\end{aligned} \tag{D.14}
$$

3. $\mathscr{L}_{4,\mathrm{int}} = g\bar{\psi}\gamma^\mu\psi B_\mu$:

$$
\begin{aligned}
\sigma_{\mathrm{L}} = \frac{e^2 g^2\left(\beta^2-\lambda^2\right)}{24\pi s v\lambda^5\kappa\left(\kappa^2-1\right)} \Bigg\{ & \left[4\lambda^4+\lambda^2\left(-8\beta^2+12\beta-v^2-5\right)\right. \\
& \left.+(1-\beta)^2\left(4\beta^2-4\beta-v^2+3\right)\right]\left(\kappa+\operatorname{arccoth}(\kappa)-\kappa^2\operatorname{arccoth}(\kappa)\right)\Bigg\}
\end{aligned} \tag{D.15}
$$

$$
\begin{aligned}
\sigma_{\mathrm{T}} = \frac{e^2 g^2}{48\pi s v\lambda^5\kappa\left(\kappa^2-1\right)} \Bigg\{ & \left[4\lambda^6+\lambda^4\left(-4\beta^2+4\beta-3v^2+1\right)\right. \\
& -\lambda^2\left(4\beta^4-8\beta^3+8\beta^2-6\beta-4\beta^2v^2+6\beta v^2-3v^2+v^4\right) \\
& \left.+(1-\beta)^2\beta^2\left(4\beta^2-\beta-v^2+3\right)\right]\left(\kappa+\operatorname{arccoth}(\kappa)-\kappa^2\operatorname{arccoth}(\kappa)\right) \\
& +2\lambda^2\left(2\lambda^2-2\lambda^2v^2+4\beta^2v^2-4\beta v^2+3v^2-v^4\right)\Bigg\}
\end{aligned} \tag{D.16}
$$

4. $\mathscr{L}_{5,\mathrm{int}} = g\bar{\psi}\gamma^\mu\gamma_5\psi B_\mu$

$$
\begin{aligned}
\sigma_{\mathrm{L}} = \frac{e^2 g^2\left(\beta^2-\lambda^2\right)}{24\pi s v\beta\lambda^4\kappa\left(\kappa^2-1\right)} \Bigg\{ & \left[4\lambda^4+\lambda^2\left(-8\beta^2+12\beta-6\right)\right. \\
& \left.+(1-\beta)^2\left(4\beta^2-4\beta+2v^2\right)\right]\left(\kappa+\operatorname{arccoth}(\kappa)-\kappa^2\operatorname{arccoth}(\kappa)\right) \\
& +4\lambda^2\left(1-v^2\right)\kappa\Bigg\}
\end{aligned} \tag{D.17}
$$

$$
\begin{aligned}
\sigma_{\mathrm{T}} = \frac{e^2 g^2}{48\pi s v\lambda^5\kappa\left(\kappa^2-1\right)} \Bigg\{ & \left[4\lambda^6+\lambda^4\left(-4\beta^2+4\beta-2v^2\right)\right. \\
& -\lambda^2\left(4\beta^4-8\beta^3+8\beta^2+2v^4-4\beta^2v^2-8\beta v^2\right) \\
& \left.+(1-\beta)^2\beta^2\left(4\beta^2-2\beta+2v^2\right)\right]\left(\kappa-\operatorname{arccoth}(\kappa)+\kappa^2\operatorname{arccoth}(\kappa)\right) \\
& +\left[4\lambda^6\left(1-1v^2\right)+\lambda^4\left(8\beta-4v^4+4\beta^2v^2-12\beta^2\right)\right. \\
& \left.+4\lambda^2(1-\beta)^2\left(2\beta^2-2\beta v^2+v^4\right)\right]\frac{1-\kappa^2}{(1-\beta)^2-\lambda^2}\operatorname{arccoth}(\kappa)\Bigg\}
\end{aligned} \tag{D.18}
$$

Table D.1: $\sigma_{\mathrm{T}}\big|_{Q^2\to 0}$ **and** $\frac{\sigma_{\mathrm{L}}}{Q^2}\big|_{Q^2\to 0}$ **for the interactions considered in this paper**. m is the mass of the fermion ψ, or charged particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively. ϕ is a scalar field, B_μ is a vector field, $u = \sqrt{1-4m^2/s}$, $\rho = M/2m$, $L = \operatorname{arctanh}\left(\sqrt{1-u}\right)$, s is the energy in COM.

#	$\mathscr{L}_{\text{int}}$	$\sigma_{\mathrm{T}}\big\|_{Q^2\to 0}$	$\frac{\sigma_{\mathrm{L}}}{Q^2}\big\|_{Q^2\to 0}$
1.	$\mathscr{L}_{1,\text{int}} = g\bar{\psi}\psi\phi$	$\frac{e^2g^2(\rho^2-1)\sqrt{1-u}u^2}{8\pi m^2(\rho^2u-1)^3} - \frac{e^2g^2u(u((\rho^4-\rho^2+1)u-2)+1)}{8\pi m^2(\rho^2u-1)^3}L$	$-\frac{e^2g^2\rho^2(\rho^2-1)\sqrt{1-u}u^3}{4\pi m^4(\rho^2u-1)^5} - \frac{e^2g^2u^3(\rho^4(u-2)u+1)}{8\pi m^4(\rho^2u-1)^5}L$
2.	$\mathscr{L}_{2,\text{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi$	$\frac{e^2g^2\rho^2\sqrt{1-u}u^2}{8\pi m^2(\rho^2u-1)^3} - \frac{e^2g^2u((\rho^2-1)\rho^2u^2+1)}{8\pi m^2(\rho^2u-1)^3}L$	$-\frac{e^2g^2\rho^4\sqrt{1-u}u^3}{4\pi m^4(\rho^2u-1)^5} + \frac{e^2g^2\rho^4u^4}{4\pi m^4(\rho^2u-1)^5}L$
3.	$\mathscr{L}_{3,\text{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi$	$\frac{e^2g^2\rho^2\sqrt{1-u}u^2}{2\pi(\rho^2u-1)^3} - \frac{e^2g^2u((\rho^2-1)\rho^2u^2+1)}{2\pi(\rho^2u-1)^3}L$	$-\frac{e^2g^2\rho^4\sqrt{1-u}u^3}{\pi m^2(\rho^2u-1)^5} + \frac{e^2g^2\rho^4u^4}{\pi m^2(\rho^2u-1)^5}L$
4.	$\mathscr{L}_{4,\text{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu$	$\frac{e^2g^2u\sqrt{1-u}(\rho^4u^2+u+1)}{12\pi m^2(\rho^2u-1)^3} + \frac{e^2g^2u(u(-2\rho^4u+2\rho^2u+u-2)-2)}{12\pi m^2(\rho^2u-1)^3}L$	$-\frac{e^2g^2\sqrt{1-u}u^2(\rho^4u(u+4)+1)}{24\pi m^4(\rho^2u-1)^5} + \frac{e^2g^2u^3(\rho^4u(u+4)+1)}{24\pi m^4(\rho^2u-1)^5}L$
5.	$\mathscr{L}_{5,\text{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu$	$\frac{e^2g^2\sqrt{1-u}u(u(\rho^4u-2)+1)}{12\pi m^2(\rho^2u-1)^3} + \frac{e^2g^2u(\rho^2(u((-2\rho^4+\rho^2-2)u+6)-2)-1)}{12\pi m^2\rho^2(\rho^2u-1)^3}L$	$-\frac{e^2g^2\sqrt{1-u}u^2(\rho^4u(u+4)-6\rho^2u+1)}{24\pi m^4(\rho^2u-1)^5} - \frac{e^2g^2u^3(\rho^4(u-4)u+2\rho^2u+1)}{24\pi m^4(\rho^2u-1)^5}L$
6.	$\mathscr{L}_{6,\text{int}} = g\bar{\psi}\sigma_{\mu\nu}\psi(\partial^\mu B^\nu - \partial^\nu B^\mu)$	$\frac{2e^2g^2\sqrt{1-u}(\rho^2u(u(-2\rho^2+\rho^4u+2)+3)-1)}{3\pi(\rho^2u-1)^3} - \frac{2e^2g^2\rho^2u(u((\rho^2-2)(\rho^2+1)u+4)+1)}{3\pi(\rho^2u-1)^3}L$	$-\frac{2e^2g^2\rho^2\sqrt{1-u}u^2(\rho^4u(u+2)+2\rho^2u+1)}{3\pi m^2(\rho^2u-1)^5} + \frac{2e^2g^2\rho^2u^3(\rho^4u(u+2)+2\rho^2u+1)}{3\pi m^2(\rho^2u-1)^5}L$
7.	$\mathscr{L}_{7,\text{int}} = g\pi^\dagger\pi\phi$	$\frac{e^2g^2\sqrt{1-u}u^2}{64\pi m^4(\rho^2u-1)^3} + \frac{e^2g^2(u-2)u^2}{64\pi m^4(\rho^2u-1)^3}L$	$-\frac{e^2g^2\sqrt{1-u}u^2(\rho^2u+1)^2}{128\pi m^6(\rho^2u-1)^5} + \frac{e^2g^2u^3(\rho^2u+1)^2}{128\pi m^6(\rho^2u-1)^5}L$
8.	$\mathscr{L}_{8,\text{int}} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu + g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	$-\frac{e^2g^2\sqrt{1-u}u(\rho^4u^2-3\rho^2u+u+1)}{48\pi m^2(\rho^2u-1)^3} + \frac{e^2g^2(\rho^2-1)(u-2)u^2}{48\pi m^2(\rho^2u-1)^3}L$	$-\frac{e^2g^2\sqrt{1-u}u^2(\rho^2(u(\rho^2((\rho^2-3)u+2)+2)+1)-3)}{96\pi m^4(\rho^2u-1)^5} - \frac{e^2g^2u^2(\rho^2u+1)(\rho^4u^2+\rho^2(u-5)u+u+2)}{96\pi m^4(\rho^2u-1)^5}L$
a.	$\mathscr{L}_{\text{a,int}} = g\bar{\psi}\psi\phi^2$	0	0
b.	$\mathscr{L}_{\text{b,int}} = g\bar{\psi}\psi B_\mu B^\mu$	0	0
c.	$\mathscr{L}_{\text{c,int}} = g\pi^\dagger\pi\phi^2$	0	0
d.	$\mathscr{L}_{\text{d,int}} = g\pi^\dagger\pi B_\mu B^\mu$	0	0

Table D.2: $\frac{\sigma_{\rm LL}}{Q^2}\big|_{Q^2\to 0}$ **and** $\frac{\sigma_{\rm TL}}{Q^2}\big|_{Q^2\to 0}$ **for the photo-disintegration of a vector particle**. m is the mass of the fermion ψ, or charged particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively. B_μ is a vector field, $u = \sqrt{1-4m^2/s}$, $\rho = M/2m$, $L = \mathrm{arctanh}\left(\sqrt{1-u}\right)$, s is the energy in COM.

#	$\mathscr{L}_{\rm int}$	$\frac{\sigma_{\rm TL}}{Q^2}\Big\|_{Q^2\to 0}$	$\frac{\sigma_{\rm LL}}{Q^2}\Big\|_{Q^2\to 0}$
1.	$\mathscr{L}_{5,\rm int} = g\bar{\psi}\gamma_\mu\psi B_\mu$	$-\frac{e^2g^2\rho^2\sqrt{1-u}u^3}{2\pi m^4(1-\rho^2u)^5} + \frac{e^2g^2\rho^2u^3}{2\pi m^4(1-\rho^2u)^5}L$	$\frac{e^2g^2\sqrt{1-u}u^2(\rho^4u(u+4)+4\rho^2u+1)}{16\pi m^4(1-\rho^2u)^5} - \frac{e^2g^2u^3(4\rho^2+\rho^4u(u+4)+1)}{16\pi m^4(1-\rho^2u)^5}L$
2.	$\mathscr{L}_{8,\rm int} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu + g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	$\frac{e^2g^2\rho^2\sqrt{1-u}u^2(u(2\rho^2+\rho^4u+8)+1)}{32\pi m^4(1-\rho^2u)^5} - \frac{e^2g^2\rho^2u^3(\rho^2u(6-\rho^2u)+7)}{32\pi m^4(1-\rho^2u)^5}L$	$-\frac{3e^2g^2\sqrt{1-u}u^2(\rho^2u+1)^2}{64\pi m^4(1-\rho^2u)^5} + \frac{e^2g^2u^2(u+2)(\rho^2u+1)^2}{64\pi m^4(1-\rho^2u)^5}L$

D.2 Forward Doubly-Virtual Compton Scattering

To preform the analysis of the Baldin-SR and LSR, only the FVVCS process has to be considered, i.e. $t = 0$. As an example, Fig. D.4 depicts the Feynman diagrams that describe, at leading order, the process $\gamma + \phi \to \gamma + \phi$, governed by $\mathscr{L}_2$. In Appendix D.3 is proved that the sum of these diagrams has electro-magnetic gauge invariance. Thus, the matrix element of Fig. D.4 can be written in a gauge invariant way as [21]

$$\mathscr{M}^{\mu\nu} = \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)T_1(\nu,Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right)T_2(\nu,Q^2), \tag{D.19}$$

where $Q^2 = -q^2$, $\nu = \frac{p\cdot q}{M}$, and $p^2 = M$. $T_1(\nu,Q^2)$ and $T_2(\nu,Q^2)$ are scalar functions that depend only on the variables ν and Q^2. Note that, for a vector particle, the polarizations can be averaged out by using the projector

$$P^{\sigma\tau} = \frac{1}{3}\left(-g^{\sigma\tau} + \frac{p^\sigma p^\tau}{M^2}\right). \tag{D.20}$$

Chapter 3 makes use of the longitudinal amplitude

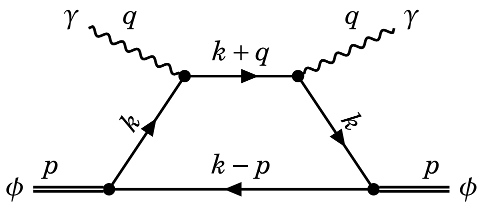
$$T_{\rm L}(\nu,Q^2) = \epsilon_{{\rm L}\mu}\epsilon'_{{\rm L}\nu}\mathscr{M}^{\mu\nu}. \tag{D.21}$$

The amplitude $T_{\rm L}(\nu,Q^2)$ can be expressed with the help of the scalar functions $T_1(\nu,Q^2)$ and $T_2(\nu,Q^2)$, and the quantities ν, Q^2, M^2. For this consider RHS of Eq. D.21, with the four-momentum p of the particles and the longitudinal polarizations $\epsilon_{\rm L} = \epsilon'_{\rm L}$ of the photons written in the LRS

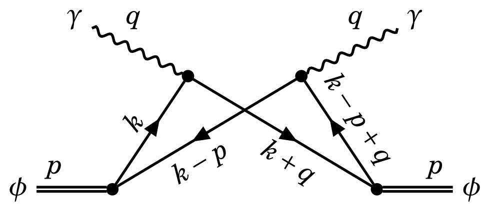
$$p = \begin{bmatrix} M \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \epsilon_{\rm L} = \epsilon'_{\rm L} = -\frac{1}{\sqrt{Q^2}}\begin{bmatrix} \sqrt{\nu^2+Q^2} \\ 0 \\ 0 \\ \nu \end{bmatrix}. \tag{D.22}$$

By replacing Eq. D.22 in Eq. D.21, $T_{\rm L}(\nu,Q^2)$ can be rewritten as

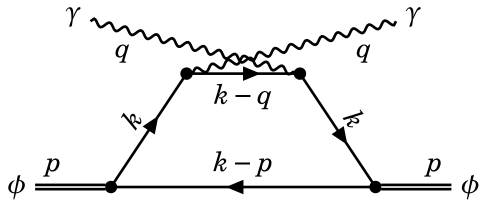
$$\begin{aligned} T_{\rm L}(\nu,Q^2) &= \epsilon_{{\rm L}\mu}\epsilon'_{{\rm L}\nu}\mathscr{M}^{\mu\nu} \\ &= -\epsilon_{\rm L}\cdot\epsilon_{\rm L}\,T_1(\nu,Q^2) + (p\cdot\epsilon_{\rm L})^2\frac{1}{M^2}T_2(\nu,Q^2) \\ &= -T_1(\nu,Q^2) + \frac{\nu^2+Q^2}{Q^2}T_2(\nu,Q^2). \end{aligned} \tag{D.23}$$



(a) Bag diagram



(b) Cat ears diagram



(c) Crossed bag diagram

Figure D.4: **Feynman diagrams of the FVVC photo-disintegration process corresponding to $\mathscr{L}_2$.** Leading FVVCS is achieved through a one-loop diagram. The incoming momenta for each particle are equal to the outgoing momenta, respectively. q is the momentum of the photon γ, p is the momentum of the scalar ϕ, and k is the unbounded momentum of the one-loop.

D.3 Checking Gauge Invariance

Gauge invariance is a redundancy of embedding particles with spin one, or higher, into a local Lagrangian [18, p. 130]. In QED, a immediate consequence of gauge invariance is the Ward identity [17, p. 348]: given $\mathscr{M}(q) = \epsilon_\mu \mathscr{M}^\mu$ be the matrix element of a Feynman diagram, where ϵ_μ is the polarization of a spin one particle with momentum p, then $p_\mu \mathscr{M}^\mu = 0$.Ward identity holds at any order in perturbation theory if and only of all the diagrams at that order are included.

In the following, only the electro-magnetic gauge invariance for the diagrams in Fig. D.4 is proven. This set of diagrams corresponds to the FVVCS photo-disintegration process of

$$\mathscr{L}_2 = \bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu)\psi + \mathrm{i}g\bar{\psi}\gamma_5\psi\phi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2.$$

In the other cases, the method is similar. The proof is based mostly on the identity $\slashed{q} = (\slashed{k} + \slashed{q} - m^2) - (\slashed{k} - m^2)$. The contractions of q_ν with the matrix elements of Fig. D.4a and Fig. D.4b are

$$\begin{aligned}
\mathrm{i}q_\nu \mathscr{M}_\mathrm{a}^{\mu\nu} &= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\gamma^5 \frac{1}{\slashed{k} - m}\slashed{q}\frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right] \\
&= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\gamma^5 \frac{1}{\slashed{k} - m}\left\{(\slashed{k} + \slashed{q} - m^2) - (\slashed{k} - m^2)\right\}\frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right] \\
&= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\gamma^5 \frac{1}{\slashed{k} - m}\gamma^\mu \frac{1}{\slashed{k} - m} - \gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\gamma^5 \frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{i}q_\nu \mathscr{M}_\mathrm{b}^{\mu\nu} &= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\slashed{q}\frac{1}{\slashed{k} - \slashed{p} + \slashed{q} - m}\gamma^5 \frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right] \\
&= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\left\{(\slashed{k} - \slashed{p} + \slashed{q} - m^2) - (\slashed{k} - \slashed{p} - m^2)\right\}\frac{1}{\slashed{k} - \slashed{p} + \slashed{q} - m}\gamma^5 \frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right] \\
&= e^2 g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 \frac{1}{\slashed{k} - \slashed{p} - m}\gamma^5 \frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m} - \gamma^5 \frac{1}{\slashed{k} - \slashed{p} + \slashed{q} - m}\gamma^5 \frac{1}{\slashed{k} + \slashed{q} - m}\gamma^\mu \frac{1}{\slashed{k} - m}\right].
\end{aligned}$$

By summing $\mathrm{i}q_\nu \mathscr{M}_\mathrm{a}^{\mu\nu}$ and $\mathrm{i}q_\nu \mathscr{M}_\mathrm{b}^{\mu\nu}$,

$$\begin{aligned}
\mathrm{i}q_\nu\mathcal{M}_{\mathrm{a}}^{\mu\nu}+\mathrm{i}q_\nu\mathcal{M}_{\mathrm{b}}^{\mu\nu} &= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\mu\frac{1}{\not k-m}-\gamma^5\frac{1}{\not k-\not p+\not q-m}\gamma^5\frac{1}{\not k+\not q-m}\gamma^\mu\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\mu\frac{1}{\not k-m}-\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\mu\frac{1}{\not k-\not q-m}\right]\\
&= -e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\mu\frac{1}{\not k-\not q-m}\not q\frac{1}{\not k-m}\right]\\
&= -\mathrm{i}q_\nu\mathcal{M}_{\mathrm{c}}^{\nu\mu}=-\mathrm{i}q_\nu\mathcal{M}_{\mathrm{c}}^{\mu\nu},
\end{aligned}$$

the electro-magnetic gauge invariance is obtained, $q_\nu\mathcal{M}_{\mathrm{a}}^{\mu\nu}+q_\nu\mathcal{M}_{\mathrm{b}}^{\mu\nu}+q_\nu\mathcal{M}_{\mathrm{c}}^{\mu\nu}=0$. The same approach is used for q_μ,

$$\begin{aligned}
\mathrm{i}q_\mu\mathcal{M}_{\mathrm{a}}^{\mu\nu} &= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k+\not q-m}\not q\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k+\not q-m}\left\{(\not k+\not q-m^2)-(\not k-m^2)\right\}\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k-m}-\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k+\not q-m}\gamma^\nu\frac{1}{\not k-m}\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{i}q_\mu\mathcal{M}_{\mathrm{b}}^{\mu\nu} &= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^\nu\frac{1}{\not k-\not p+\not q-m}\gamma^5\frac{1}{\not k+\not q-m}\not q\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k+\not q-m}\not q\frac{1}{\not k-m}\gamma^5\frac{1}{\not k-\not p-m}\gamma^\nu\frac{1}{\not k-\not p+\not q-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k+\not p-m}\not q\frac{1}{\not k+\not p-\not q-m}\gamma^5\frac{1}{\not k-\not q-m}\gamma^\nu\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k+\not p-m}\left\{(\not k+\not p-m^2)-(\not k+\not p-\not q-m^2)\right\}\frac{1}{\not k+\not p-\not q-m}\gamma^5\frac{1}{\not k-\not q-m}\gamma^\nu\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k+\not p-\not q-m}\gamma^5\frac{1}{\not k-\not q-m}\gamma^\nu\frac{1}{\not k-m}-\gamma^5\frac{1}{\not k+\not p-m}\gamma^5\frac{1}{\not k-\not q-m}\gamma^\nu\frac{1}{\not k-m}\right]\\
&= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k+\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k+\not q-m}-\gamma^5\frac{1}{\not k+\not p-m}\gamma^5\frac{1}{\not k-\not q-m}\gamma^\nu\frac{1}{\not k-m}\right].
\end{aligned}$$

By making change of variable $k\to-k$ and using the fact that $\mathrm{Tr}\left[\gamma^5(\not a-m)\gamma^5(\not b-m)\gamma^\nu(\not c-m)\right]=\mathrm{Tr}\left[\gamma^5(\not a+m)\gamma^5(\not b+m)\gamma^\nu(\not c+m)\right]$, $\mathrm{i}q_\mu\mathcal{M}_{\mathrm{b}}^{\mu\nu}$ can be rewritten as

$$\mathrm{i}q_\mu\mathcal{M}_{\mathrm{b}}^{\mu\nu}=e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[-\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k-\not q-m}+\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k+\not q-m}\gamma^\nu\frac{1}{\not k-m}\right].$$

As above:

$$\begin{aligned}
\mathrm{i}q_\mu\mathcal{M}_{\mathrm{a}}^{\mu\nu}+\mathrm{i}q_\mu\mathcal{M}_{\mathrm{b}}^{\mu\nu} &= e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k-m}-\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k-\not q-m}\right]\\
&= -e^2g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^5\frac{1}{\not k-\not p-m}\gamma^5\frac{1}{\not k-m}\gamma^\nu\frac{1}{\not k-\not q-m}\not q\frac{1}{\not k-m}\right]\\
&= -\mathrm{i}q_\mu\mathcal{M}_{\mathrm{c}}^{\mu\nu},
\end{aligned}$$

thus the relation $q_\nu\mathcal{M}_{\mathrm{a}}^{\mu\nu}+q_\nu\mathcal{M}_{\mathrm{b}}^{\mu\nu}+q_\nu\mathcal{M}_{\mathrm{c}}^{\mu\nu}=0$ holds.

Appendix E

Baldin Sum-Rule and Longitudinal Sum-Rule

E.1 Derivation of the Baldin Sum-Rule

Consider a neutral particle that has no spin, characterized by electric and magnetic dipole polarizabilities α_{E1}, and β_{M1}, respectively. In NRQM, such a particle can be described by using the Hamiltonian

$$\mathcal{H} = \frac{1}{2M}\mathbf{p}^2 - \frac{1}{2}\alpha_{\mathrm{E1}}\mathbf{E}^2 - \frac{1}{2}\beta_{\mathrm{M1}}\mathbf{B}^2, \tag{E.1}$$

with M the particle mass, $\mathbf{E}$ an external electric field, and $\mathbf{B}$ an external magnetic field. Accordingly, the Compton scattering matrix element in the forward direction is given by [28]

$$\mathcal{M}_{\mathrm{NRQM}} = \epsilon\cdot\epsilon'(\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}})\nu^2, \tag{E.2}$$

where ϵ and ϵ' are the incoming and outgoing photon polarizations, respectively, and ν is the photon energy.

The matrix element of FVVCS with on-shell photons has the form

$$\mathcal{M} = \epsilon\cdot\epsilon' T_1(\nu). \tag{E.3}$$

Note that $T_1(\nu) = T_1(\nu, Q^2 = 0)$. By expanding T_1, in the above relation, up to second order in the photon energy ν and by using the low energy theorems in [3; 4; 29],

$$\begin{aligned} T_1(\nu)\big|_{\nu=0} &= 0 + \underbrace{C_{T_1}}_{\alpha_{\mathrm{E1}}+\beta_{\mathrm{M1}}}\nu^2 + \mathcal{O}(\nu^4) \\ \mathcal{M} &= \epsilon\cdot\epsilon'\left(0 + C_{T_1}\nu^2\right) + \mathcal{O}(\nu^4). \end{aligned} \tag{E.4}$$

The coefficient C_{T_1} can be identified with $\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}}$ in Eq. E.2: $C_{T_1} = \alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}}$.

The zero in Eq. E.4 implies that T_1 can be written using a once-subtracted dispersion relation with $T_1(\nu = 0) = 0$ and

$$\begin{aligned} T_1(\nu) &= 0 + \frac{2\nu^2}{\pi}\int_0^\infty \mathrm{d}\nu' \,\frac{\mathrm{Im}\, T_1(\nu')}{\nu'(\nu'^2 - \nu^2)}, \\ &= 0 + \frac{4M\nu^2}{\pi}\int_0^\infty \mathrm{d}\nu'\, \frac{\sigma(\nu')}{\nu'^2 - \nu^2}. \end{aligned} \tag{E.5}$$

$\sigma(\nu') = \sigma_{\mathrm{T}}(\nu', Q^2 = 0)$ is the photo-absorption cross-section of the photo-disintegration process in the on-sell case as a function of the photon energy in LRS, i.e. $\nu \to \nu'$. The above equation can be also expanded in ν around zero up to second-order, with the result

$$T_1(\nu)\big|_{\nu=0} = \nu^2 \underbrace{\frac{4M}{\pi}\int_0^\infty \mathrm{d}\nu' \frac{\sigma(\nu')}{\nu'^2}}_{\alpha_{\mathrm{E1}}+\beta_{\mathrm{M1}}} + \mathcal{O}(\nu^4). \tag{E.6}$$

Equations E.2, E.4 and E.6, yield the Baldin-SR [5]

$$\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}} = \frac{4M}{\pi}\int_{\nu_0}^\infty \mathrm{d}\nu' \frac{\sigma(\nu')}{\nu'^2}, \tag{E.7}$$

where ν represents the photon energy in the LRS. The cross-section $\sigma(\nu)$ is zero up to the threshold ν_0 for the photo-disintegration process. This is the reason why the lower integration limit is ν_0 instead of 0.

To investigate if the Baldin-SR holds, the knowledge of $\sigma(\nu)$ and of the coefficient C_{T_1} is required. $\sigma(\nu)$ is given by the on-shell photo-absorption cross-sections determined in Appendix D.1.1, i.e. $\sigma(\nu) = \sigma_{\mathrm{T}}(\nu, Q^2 = 0)$. C_{T_1} can be computed by expanding $T_1(\nu, 0)$ in ν around $\nu = 0$ to second order. However, Eq. E.8 offers another possibility to calculate C_{T_1} [13]:

$$T_1(\nu,0) = \nu^2 \lim_{Q^2\to 0} \frac{1}{Q^2} T_2(\nu,Q^2) \quad \Longrightarrow \quad C_{\mathrm{T}_1} = \lim_{Q^2\to 0}\frac{1}{Q^2} T_2(0,Q^2). \tag{E.8}$$

This equation follows from the fact that the longitudinal amplitude

$$T_{\mathrm{L}}(\nu,Q^2) = -T_1(\nu,Q^2) + \frac{\nu^2+Q^2}{Q^2} T_2(\nu,Q^2), \tag{E.9}$$

and the scalar function $T_2(\nu,Q^2)$ are null for on-shell photons

$$\begin{aligned} &\lim_{Q^2\to 0} T_{\mathrm{L}}(\nu,Q^2) = 0, \\ &\lim_{Q^2\to 0} T_2(\nu,Q^2) = 0. \end{aligned} \tag{E.10}$$

Consequently, the Baldin-SR can be rewritten as

$$\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}} = \lim_{Q^2\to 0}\frac{1}{Q^2} T_2(0,Q^2) = \frac{4M}{\pi}\int_{\nu_0}^\infty \mathrm{d}\nu' \frac{\sigma(\nu')}{\nu'^2}. \tag{E.11}$$

E.2 Baldin Sum-Rule Results

Table E.1 contains the results of the Baldin-SR for the interaction Lagrangians analyzed in the present paper. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively.

E.3 Derivation of the Longitudinal Sum-Rule

By considering again the NRQM Hamiltonian of a neutral particle that has no spin, see Eq. E.1,

$$\mathcal{H} = \frac{1}{2M}\mathbf{p}^2 - \frac{1}{2}\alpha_{\mathrm{E1}}\mathbf{E}^2 - \frac{1}{2}\beta_{\mathrm{M1}}\mathbf{B}^2, \tag{E.12}$$

the electric polarizability can be recovered from the $\mathcal{M}^{00}$ component of the on-shell matrix element [14]

$$\mathcal{M}^{00}(\nu)\big|_{\nu=0} = \alpha_{\mathrm{E1}}\nu^2 + \mathcal{O}(\nu^4). \tag{E.13}$$

Table E.1: $\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}}$ **for the interactions considered in this paper**. m is the mass of the fermion ψ, or charged particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively. ϕ is a scalar field, B_μ is a vector field, and $\rho = M/2m$.

#	$\mathcal{L}_{\mathrm{int}}$	$\alpha_{\mathrm{E1}} + \beta_{\mathrm{M1}}$
1.	$\mathcal{L}_{1,\mathrm{int}} = g\bar{\psi}\psi\phi$	$\frac{e^2g^2(2\rho^2+1)}{\pi^2m^2 16\rho^2(1-16\rho^2)} - \frac{e^2g^2(-16\rho^4+4\rho^2+3)\arcsin(\rho)}{48\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$
2.	$\mathcal{L}_{2,\mathrm{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi$	$\frac{e^2g^2(3-2\rho^2)}{16\pi^2m^2(\rho^2-1)^2} + \frac{e^2g^2(-16\rho^4+28\rho^2-9)\arcsin(\rho)}{48\pi^2m^2\rho(1-\rho^2)^{5/2}}$
3.	$\mathcal{L}_{3,\mathrm{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi$	$\frac{e^2g^2(3-2\rho^2)}{4\pi^2(\rho^2-1)^2} + \frac{e^2g^2(-16\rho^4+28\rho^2-9)\arcsin(\rho)}{12\pi^2\rho(1-\rho^2)^{5/2}}$
4.	$\mathcal{L}_{4,\mathrm{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu$	$\frac{e^2g^2(-12\rho^4+28\rho^2-13)}{48\pi^2m^2\rho^2(\rho^2-1)^2} + \frac{e^2g^2(-32\rho^6+112\rho^4-110\rho^2+39)\arcsin(\rho)}{144\pi^2m^2\rho^3(1-\rho^2)^{5/2}}$
5.	$\mathcal{L}_{5,\mathrm{int}} = g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu$	$\frac{e^2g^2(1-2\rho^2)}{8\pi^2m^2\rho^2(\rho^2-1)} + \frac{e^2g^2(16\rho^4-16\rho^2+9)\arcsin(\rho)}{72\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$
6.	$\mathcal{L}_{6,\mathrm{int}} = g\bar{\psi}\sigma_{\mu\nu}\psi(\partial^\mu B^\nu - \partial^\nu B^\mu)$	$\frac{e^2g^2(-6\rho^4+13\rho^2-6)}{\pi^2(\rho^2-1)^2} + \frac{e^2g^2(-16\rho^6+92\rho^4-121\rho^2+54)\arcsin(\rho)}{9\pi^2\rho(1-\rho^2)^{5/2}}$
7.	$\mathcal{L}_{7,\mathrm{int}} = g\pi^\dagger\pi\phi$	$\frac{e^2g^2(2\rho^2-1)}{128\pi^2m^4\rho^2(\rho^2-1)^2} + \frac{e^2g^2(8\rho^4-8\rho^2+3)\arcsin(\rho)}{384\pi^2m^4\rho^3(1-\rho^2)^{5/2}}$
8.	$\mathcal{L}_{8,\mathrm{int}} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu + g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	$\frac{e^2g^2(13-10\rho^2)}{96\pi^2m^2\rho^2(1-\rho^2)} - \frac{e^2g^2(8(\rho^2-7)\rho^2+39)\arcsin(\rho)}{288\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$
a.	$\mathcal{L}_{\mathrm{a,int}} = g\bar{\psi}\psi\phi^2$	0
b.	$\mathcal{L}_{\mathrm{b,int}} = \mathrm{i}g\bar{\psi}\gamma_5\psi B_\mu B^\mu$	0
c.	$\mathcal{L}_{\mathrm{c,int}} = g\pi^\dagger\pi\phi^2$	0
d.	$\mathcal{L}_{\mathrm{d,int}} = g\pi^\dagger\pi B_\mu B^\mu$	0

In the QED case, the same relation holds if $\mathcal{M}^{00}(\nu, Q^2 = 0)$ of the FVVCS process is expanded in $\nu = 0$ to second order.

The amplitude $\mathcal{M}^{00}(\nu, Q^2)$ is linked to the FVVCS longitudinal amplitude $T_{\mathrm{L}}(\nu, Q^2) = \epsilon_{\mathrm{L}\mu}\epsilon'_{\mathrm{L}\nu}\mathcal{M}^{\mu\nu}$. To establish the linkage, consider the LRF, where $\epsilon_{\mathrm{L}\mu}$ can be taken as

$$\epsilon_{\mathrm{L}\mu} = \begin{bmatrix} \sqrt{\nu^2+Q^2} \\ 0 \\ 0 \\ \nu \end{bmatrix}, \tag{E.14}$$

ν being the photon energy in LRF. A gauge transformation with the parameter

$$\lambda = \frac{\nu}{\sqrt{Q^2\left(Q^2+\nu^2\right)}} \tag{E.15}$$

eliminates the spatial component of $\epsilon_{\mathrm{L}\mu}$

$$\epsilon_{\mathrm{L}\mu} - \lambda q_\mu = -\frac{1}{\sqrt{Q^2}} \begin{bmatrix} \sqrt{\nu^2+Q^2} \\ 0 \\ 0 \\ \nu \end{bmatrix} - \frac{\nu}{\sqrt{Q^2\left(Q^2+\nu^2\right)}} \begin{bmatrix} \nu \\ 0 \\ 0 \\ \sqrt{\nu^2+Q^2} \end{bmatrix} = \sqrt{\frac{Q^2}{\nu^2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{\frac{Q^2}{\nu^2}}\bar{\epsilon}_{\mathrm{L}\mu}. \tag{E.16}$$

In the FVVCS, the relations $\epsilon_{L\mu} = \epsilon_{L\nu}$, and $\bar{\epsilon}_{L\mu} = \bar{\epsilon}'_{L\nu}$ hold. As a consequence,

$$
\begin{aligned}
T_L(\nu,Q^2) &= \epsilon_{L\mu}\epsilon'_{L\nu}\mathcal{M}^{\mu\nu}(\nu,Q^2) \\
&= \frac{\nu^2}{Q^2}\bar{\epsilon}_{L\mu}\bar{\epsilon}_{L\nu}\mathcal{M}^{\mu\nu}(\nu,Q^2) \\
&= \frac{\nu^2}{Q^2}\mathcal{M}^{00}(\nu,Q^2) \\
&\Downarrow \\
\mathcal{M}^{00}(\nu,Q^2) &= \frac{\nu^2}{Q^2}T_L(\nu,Q^2).
\end{aligned}
\tag{E.17}
$$

By considering the last relation on-shell, and expanding in ν around zero up to second order, the connection between $T_L(\nu,Q^2)$ and the electric polarizability α_{E1} can be established,

$$
\mathcal{M}^{00}(\nu,Q^2) = \frac{\nu^2}{Q^2}T_L(\nu,Q^2) \Longrightarrow \mathcal{M}^{00}(\nu,0)\big|_{\nu=0} = \nu^2 \underbrace{\lim_{Q^2\to 0}\frac{1}{Q^2}T_L(0,Q^2)}_{\alpha_{E1}} + \mathcal{O}\left(\nu^4\right).
\tag{E.18}
$$

Presume that $\nu^{-2}\mathcal{M}^{00}(\nu,Q^2)$ can be written with the help of an unsubstracted dispersion relation, see Eq. E.19. By using Eq. E.17, and the optical theorem for $T_L(\nu,Q^2) = 2\nu M\sigma_L(\nu,Q^2)$, the unsubstracted dispersion relation for $\nu^{-2}\mathcal{M}^{00}$ can be brought to the from in Eq. E.20.

$$
\operatorname{Re}\frac{1}{\nu^2}\mathcal{M}^{00}(\nu,Q^2) = \frac{2}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{\nu'}{\nu'^2-\nu^2}\operatorname{Im}\frac{1}{\nu'^2}\mathcal{M}^{00}(\nu',Q^2)
\tag{E.19}
$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{\nu'}{\nu'^2-\nu^2}\operatorname{Im}\frac{1}{Q^2}T_L(\nu',Q^2) \\
&= \frac{4M}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{\nu'^2}{\nu'^2-\nu^2}\frac{\sigma_L(\nu',Q^2)}{Q^2}.
\end{aligned}
\tag{E.20}
$$

The on-shell expansion of Eq. E.20 in $\nu = 0$ to nullth order yields

$$
\mathcal{M}^{00}(\nu,0)\big|_{\nu=0} = \nu^2 \underbrace{\frac{4M}{\pi}\int_{\nu_0}^{\infty} d\nu' \lim_{Q^2\to 0}\frac{\sigma_L(\nu',Q^2)}{Q^2}}_{\alpha_{E1}} + \mathcal{O}\left(\nu^4\right),
\tag{E.21}
$$

where ν_0 is the photon-energy threshold of the photo-disintegration. Note that at $\nu = 0$, and also below the threshold ν_0, $\mathcal{M}^{00}(\nu,0)$ has no imaginary part, see the assumptions made in Chapter 2.2.

E.4 Longitudinal Sum-Rule Results

The quantities $\frac{T_L(\nu,Q^2)}{Q^2}$ and Δ_L in the case $\nu \to 0$ and $Q^2 \to 0$ are listed in Table E.2. The following two expressions illustrate the from of $\frac{T_L(\nu,Q^2)}{Q^2}$ for any ν when the interactions $\mathcal{L}_{1,\text{int}}$ and $\mathcal{L}_{4,\text{int}}$ are considered. $\rho = \frac{M}{2m}$, $\rho_p = \frac{M+2\nu}{2m}$ and $\rho_p = \frac{M-2\nu}{2m}$.

1. $\mathcal{L}_{1,\text{int}} = g\pi^\dagger\pi\phi$:

$$
\frac{T_L(\nu,Q^2)}{Q^2}\Big|_{Q^2\to 0} = e^2g^2\frac{3}{8\pi^2m^2\nu^2}
$$

$$
\begin{aligned}
&- e^2 g^2 \frac{4m^2\rho^2\left(\rho^2-1\right)+\left(5\rho^2-4\right)\nu^2}{8\pi^2 m^2 \rho\sqrt{1-\rho^2}\nu^4}\arcsin\left(\sqrt{\rho}\right)\\
&+ e^2 g^2 \frac{4m^2\rho^2+\nu^2}{16\pi^2 m^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho}\right)\\
&- e^2 g^2 \frac{\sqrt{\rho_m(1-\rho\rho_m)}(\nu-2m\rho)^2}{16\pi^2 m^2\rho^{3/2}\nu^4}\arcsin\left(\sqrt{\rho\rho_m}\right)\\
&- e^2 g^2 \frac{(\nu-2m\rho)^2}{32\pi^2 m^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho\rho_m}\right)\\
&- e^2 g^2 \frac{\sqrt{\rho_p(1-\rho\rho_p)}(2m\rho+\nu)^2}{16\pi^2 m^2\rho^{3/2}\nu^4}\arcsin\left(\sqrt{\rho\rho_p}\right)\\
&- e^2 g^2 \frac{(\nu+2m\rho)^2}{32\pi^2 m^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho\rho_p}\right) \qquad \text{(E.22)}
\end{aligned}
$$

2. $\mathscr{L}_{4,\text{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu$:

$$
\begin{aligned}
\left.\frac{T_\text{L}(\nu,Q^2)}{Q^2}\right|_{Q^2\to 0} = &-e^2 g^2 \frac{1}{9\pi^2 m^2} + e^2 g^2 \frac{\left(2\rho^2+1\right)}{\pi^2\nu^2}\\
&+ e^2 g^2 \frac{4m^2\rho^2\left(-2\rho^4+\rho^2+1\right)+\left(5-8\rho^4\right)\nu^2}{3\pi^2\rho\sqrt{1-\rho^2}\nu^4}\arcsin\left(\sqrt{\rho}\right)\\
&- e^2 g^2 \frac{2m^2\rho^2\left(2\rho^2+1\right)+\nu^2}{3\pi^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho}\right)\\
&- e^2 g^2 \frac{e^2 g^2\sqrt{\rho_m(1-\rho\rho_m)}\left(2m^2\rho\left(2\rho^2+1\right)\rho_m+\nu^2\right)}{3\pi^2\rho^{3/2}\nu^4}\arcsin\left(\sqrt{\rho\rho_m}\right)\\
&- e^2 g^2 \frac{2m^2\rho^2(2\rho\rho_m+1)-2m\rho\nu+\nu^2}{6\pi^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho\rho_m}\right)\\
&- e^2 g^2 \frac{e^2 g^2\sqrt{\rho_p(1-\rho\rho_p)}\left(2m^2\rho\left(2\rho^2+1\right)\rho_p+\nu^2\right)}{3\pi^2\rho^{3/2}\nu^4}\arcsin\left(\sqrt{\rho\rho_p}\right)\\
&- e^2 g^2 \frac{2m^2\rho^2(2\rho\rho_p+1)+2m\rho\nu+\nu^2}{6\pi^2\rho^2\nu^4}\arcsin^2\left(\sqrt{\rho\rho_p}\right) \qquad \text{(E.23)}
\end{aligned}
$$

E.5 Closed Form of the Limit $\lim_{\nu\to\infty} T_\text{L}(\nu,Q^2)$

In Chapter 3 the quantity

$$
\Delta_\text{L} = \lim_{\substack{\nu\to\infty\\ Q^2\to 0}} \frac{1}{Q^2} T_\text{L}(\nu,Q^2) \qquad \text{(E.24)}
$$

is given for the interactions in Table 1.1. The algorithm in Appendix F.2 makes the computation of the limit $\lim_{\nu\to\infty} T_\text{L}(\nu,Q^2)$ accessible. Table E.3 lists a closed form of $\lim_{\nu\to\infty} T_\text{L}(\nu,Q^2)$ for each of the interactions in Table 1.1, accompanied by the limit in the cases $Q^2\to 0$, and $Q^2\to\infty$.

Table E.2: $\frac{T_{\rm L}(\nu,Q^2)}{Q^2}$ **and** $\Delta_{\rm L}$ **in the case** $(\nu\to 0, Q^2\to 0)$**, for the interactions considered in this paper**. m is the mass of the fermion ψ , or charged particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively. ϕ is a scalar field, B_μ is a vector field, and $\rho = M/2m$. The last column represents the dimension of the interaction Lagrangian without counting the dimension of the couplings.

#	$\mathscr{L}_{\rm int}$	$\left.\frac{T_{\rm L}(\nu,Q^2)}{Q^2}\right\vert_{\nu\to0,\, Q^2\to0}$	$\Delta_{\rm L}$	$[\mathscr{L}_{\rm int}]_{\rm fields}$
1.	$\mathscr{L}_{1,\rm int} = g\bar\psi\psi\phi$	$\frac{e^2g^2(2\rho^2+1)}{16\pi^2m^2\rho^2(1-\rho^2)} + \frac{e^2g^2(8\rho^4+4\rho^2-3)\arcsin(\rho)}{48\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$	$-\frac{e^2g^2}{6\pi^2m^2}$	4
2.	$\mathscr{L}_{2,\rm int} = {\rm i}g\bar\psi\psi\gamma_5\phi$	$\frac{e^2g^2(2\rho^2+1)}{48\pi^2m^2(\rho^2-1)^2} - \frac{e^2g^2(8\rho^4-20\rho^2+9)\arcsin(\rho)}{48\pi^2m^2\rho(1-\rho^2)^{5/2}}$	$-\frac{e^2g^2}{6\pi^2m^2}$	4
3.	$\mathscr{L}_{3,\rm int} = g\bar\psi\gamma_\mu\gamma_5\psi\partial^\mu\phi$	$\frac{e^2g^2(8\rho^4-14\rho^2+9)}{12\pi^2(\rho^2-1)^2} - \frac{e^2g^2(8\rho^4-20\rho^2+9)\arcsin(\rho)}{12\pi^2\rho(1-\rho^2)^{5/2}}$	0	5
4.	$\mathscr{L}_{4,\rm int} = g\bar\psi\gamma_\mu\psi B_\mu$	$-\frac{e^2g^2(2\rho^2-5)(2\rho^2-1)}{48\pi^2m^2\rho^2(\rho^2-1)^2} - \frac{e^2g^2(2\rho^2-5)(8\rho^4-8\rho^2+3)\arcsin(\rho)}{144\pi^2m^2\rho^3(1-\rho^2)^{5/2}}$	$-\frac{e^2g^2}{9\pi^2m^2}$	4
5.	$\mathscr{L}_{5,\rm int} = g\bar\psi\gamma_\mu\gamma_5\psi B_\mu$	$\frac{e^2g^2(1-2\rho^2)}{8\pi^2m^2\rho^2(\rho^2-1)} + \frac{e^2g^2(8\rho^4-8\rho^2+9)\arcsin(\rho)}{72\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$	$-\frac{e^2g^2}{9\pi^2m^2}$	4
6.	$\mathscr{L}_{6,\rm int} = g\bar\psi\sigma_{\mu\nu}\psi(\partial^\mu B^\nu - \partial^\nu B^\mu)$	$\frac{e^2gm(-8\rho^6+28\rho^4-17\rho^2+6)\arcsin(\rho)}{6\pi^2\rho(1-\rho^2)^{5/2}} - \frac{e^2gm(2\rho^4-7\rho^2+2)}{2\pi^2(\rho^2-1)^2}$	$-\frac{2e^2g^2M^2}{9\pi^2m^2}$	5
7.	$\mathscr{L}_{7,\rm int} = g\pi^\dagger\pi\phi$	$\frac{e^2g^2(4\rho^4-4\rho^2+3)\arcsin(\rho)}{384\pi^2m^4\rho^3(1-\rho^2)^{5/2}} + \frac{e^2g^2(-4\rho^4+10\rho^2-3)}{384\pi^2m^4\rho^2(\rho^2-1)^2}$	0	3
8.	$\mathscr{L}_{8,\rm int} = {\rm i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu$ $+ g^2\pi^\dagger\pi B_\mu B^\mu$ $+ 2eg\pi^\dagger\pi A_\mu B^\mu$	$\frac{e^2g^2(4\rho^4-6\rho^2+5)}{96\pi^2m^2\rho^2(1-\rho^2)} - \frac{e^2g^2(4(\rho^2-7)\rho^2+15)\arcsin(\rho)}{288\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$	$-\frac{e^2g^2}{36\pi^2m^2}$	4
a.	$\mathscr{L}_{\rm a,int} = g\bar\psi\psi\phi^2$	$+\frac{e^2g}{3\pi^2m}$	$+\frac{e^2g}{3\pi^2m}$	4
b.	$\mathscr{L}_{\rm b,int} = g\bar\psi\psi B_\mu B^\mu$	$-\frac{e^2g}{3\pi^2m}$	$-\frac{e^2g}{3\pi^2m}$	5
c.	$\mathscr{L}_{\rm c,int} = g\pi^\dagger\pi\phi^2$	$+\frac{e^2g}{24\pi^2m^2}$	$+\frac{e^2g}{24\pi^2m^2}$	4
d.	$\mathscr{L}_{\rm d,int} = g\pi^\dagger\pi B_\mu B^\mu$	$-\frac{e^2g}{24\pi^2m^2}$	$-\frac{e^2g}{24\pi^2m^2}$	4
3'.	$\mathscr{L}'_{3,\rm int} = {\rm i}g\bar\psi\psi\gamma_5\phi$ $+ \frac{1}{2m}g^2\bar\psi\psi\phi^2$	$\frac{e^2g^2(8\rho^4-14\rho^2+9)}{48\pi^2m^2(\rho^2-1)^2} - \frac{e^2g^2(8\rho^4-20\rho^2+9)\arcsin(\rho)}{48\pi^2m^2\rho(1-\rho^2)^{5/2}}$	0	$4 \oslash 5$
4'.	$\mathscr{L}'_{4,\rm int} = g\bar\psi\gamma_\mu\psi B_\mu$ $- \frac{1}{3m}g^2\bar\psi\psi B_\mu B^\mu$	$\frac{e^2g^2(16\rho^6-44\rho^4+52\rho^2-15)}{144\pi^2m^2\rho^2(\rho^2-1)^2} - \frac{e^2g^2(2\rho^2-5)(8\rho^4-8\rho^2+3)\arcsin(\rho)}{144\pi^2m^2\rho^3(1-\rho^2)^{5/2}}$	0	$4 \oslash 5$
8'.	$\mathscr{L}'_{8,\rm int} = {\rm i}g(\pi\partial_\mu\pi^\dagger - \pi^\dagger\partial_\mu\pi)B^\mu$ $+ \frac{1}{3}g^2\pi^\dagger\pi B_\mu B^\mu$ $+ 2eg\pi^\dagger\pi A_\mu B^\mu$	$-\frac{e^2g^2(4\rho^4-10\rho^2+15)}{288\pi^2m^2\rho^2(\rho^2-1)} - \frac{e^2g^2(4(\rho^2-7)\rho^2+15)\arcsin(\rho)}{288\pi^2m^2\rho^3(1-\rho^2)^{3/2}}$	0	4

Table E.3: **Closed form of $\lim_{\nu\to\infty} T_{\rm L}(\nu,Q^2)$ for the interactions considered in this paper**. m is the mass of the fermion ψ, or charged scalar particle π. M is the mass of the incoming particle. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, respectively. ϕ is a scalar field, B_μ is a vector field, and Q^2 is the photon energy-momentum transfer.

#	$\mathscr{L}_{\rm int}$	$\lim_{\nu\to\infty} T_{\rm L}(\nu,Q^2)$	$\lim_{\substack{\nu\to\infty\\ Q\to 0}} \frac{T_{\rm L}(\nu,Q^2)}{Q^2}$	$\lim_{\substack{\nu\to\infty\\ Q\to\infty}} T_{\rm L}(\nu,Q^2)$
1.	$\mathscr{L}_{1,\rm int} = g\bar\psi\psi\phi$	$\frac{e^2g^2}{\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g^2}{6\pi^2m^2}$	$-\frac{e^2g^2}{\pi^2}$
2.	$\mathscr{L}_{2,\rm int} = \mathrm{i}g\bar\psi\psi\gamma_5\phi$	$\frac{e^2g^2}{\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g^2}{6\pi^2m^2}$	$-\frac{e^2g^2}{\pi^2}$
3.	$\mathscr{L}_{3,\rm int} = g\bar\psi\gamma_\mu\gamma_5\psi\partial^\mu\phi$	0	0	0
4.	$\mathscr{L}_{4,\rm int} = g\bar\psi\gamma_\mu\psi B_\mu$	$\frac{2e^2g^2}{3\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g^2}{9\pi^2m^2}$	$-\frac{2e^2g^2}{3\pi^2}$
5.	$\mathscr{L}_{5,\rm int} = g\bar\psi\gamma_\mu\gamma_5\psi B_\mu$	$\frac{2e^2g^2}{3\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g^2}{9\pi^2m^2}$	$-\frac{2e^2g^2}{3\pi^2}$
6.	$\mathscr{L}_{6,\rm int} = g\bar\psi\sigma_{\mu\nu}\psi(\partial^\mu B^\nu-\partial^\nu B^\mu)$	$\frac{4e^2g^2M^2}{3\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{2e^2g^2M^2}{9\pi^2m^2}$	$-\frac{4e^2g^2M^2}{3\pi^2}$
7.	$\mathscr{L}_{7,\rm int} = g\pi^\dagger\pi\phi$	0	0	0
8.	$\mathscr{L}_{8,\rm int} = \mathrm{i}g(\pi\partial_\mu\pi^\dagger-\pi^\dagger\partial_\mu\pi)B^\mu + g^2\pi^\dagger\pi B_\mu B^\mu + 2eg\pi^\dagger\pi A_\mu B^\mu$	$\frac{e^2g^2}{3\pi^2}\left[-1+\frac{\sqrt{4m^2+Q^2}}{Q}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g^2}{36\pi^2m^2}$	$-\infty$
a.	$\mathscr{L}_{a,\rm int} = g\bar\psi\psi\phi^2$	$\frac{2e^2gm}{\pi^2}\left[1-\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$+\frac{e^2g}{3\pi^2m}$	$+\frac{2e^2gm}{\pi^2}$
b.	$\mathscr{L}_{b,\rm int} = g\bar\psi\psi B_\mu B^\mu$	$\frac{2e^2g}{\pi^2}\left[-1+\frac{4m^2}{Q\sqrt{4m^2+Q^2}}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g}{3\pi^2m}$	$-\frac{2e^2gm}{\pi^2}$
c.	$\mathscr{L}_{c,\rm int} = g\pi^\dagger\pi\phi^2$	$\frac{e^2g}{2\pi^2}\left[-1+\frac{\sqrt{4m^2+Q^2}}{Q}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$+\frac{e^2g}{24\pi^2m^2}$	$+\infty$
d.	$\mathscr{L}_{d,\rm int} = g\pi^\dagger\pi B_\mu B^\mu$	$\frac{e^2g}{2\pi^2}\left[1-\frac{\sqrt{4m^2+Q^2}}{Q}\operatorname{arcsinh}\left(\frac{Q}{2m}\right)\right]$	$-\frac{e^2g}{24\pi^2m^2}$	$-\infty$

Appendix F

One-Loop Scalar Integrals

F.1 One-Loop Integrals: Definition and Direct Calculation

One-loop integrals arise in a Feynman diagram if: (i) one 4-momentum, let it be k, cannot be fixed by external momenta of the diagram and by the momentum-conservation laws at the vertices of the diagram, (ii) the edges and the vertices through which k flows do not enclose another unbounded 4-momentum of the diagram. Fig. F.1 shows a Feynman diagram that is equivalent to a one-loop integral.

One-loop integrals can be divided into two categories:

1. Scalar one-loop integrals: the numerator contains no components of the four-vector k, e.g.

$$I_{\mathrm{s}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \prod_{i=1}^{n} \frac{1}{\left[(k+q_i)^2 - m_i^2\right]}. \tag{F.1}$$

2. Tensor one-loop integrals: the numerator contains a tensor product of the components of the four-vector k, e.g.

$$I_{\mathrm{t}}^{\alpha_1\cdots\alpha_r} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} k^{\alpha_1}\cdots k^{\alpha_r} \prod_{i=1}^{n} \frac{1}{\left[(k+q_i)^2 - m_i^2\right]}. \tag{F.2}$$

One of the most used methods to calculate one-loop integrals is to firstly undertake a Feynman parametrisation. The Feynman parametrisation brings the product of therms in the denominator to a linear combination of these terms raised to a given power. Nevertheless, the Feynman parametrisation comes at the cost of introducing additional integration parameters. The master formula for the Feynman parametrisation is

$$\frac{1}{A_1^{a_1}\cdots A_m^{a_m}} = \frac{\Gamma(a_1+\cdots+a_m)}{\Gamma(a_1)\cdots\Gamma(a_m)} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_m \frac{\delta(1-x_1-\cdots-x_m)\, x_1^{a_1-1}\cdots x_m^{a_m-1}}{\left[\sum_{i=1}^m x_i A_i\right]^{\sum_{i=1}^m a_i}}. \tag{F.3}$$

$\Gamma(a)$ is the Gamma function, which becomes $\Gamma(a) = (a-1)!$ if a is a positive integer. In this thesis a variation of the above formula is used

$$\frac{1}{A_1^{a_1}\cdots A_m^{a_m}} = \frac{\Gamma(a_1+\cdots+a_m)}{\Gamma(a_1)\cdots\Gamma(a_m)} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \cdots \int_0^{1-\sum_{i=1}^{m-2} x_i} \mathrm{d}x_{m-1} \frac{x_1^{a_1-1}\cdots x_{m-1}^{a_{m-1}-1}\left(1-\sum_{i=1}^{m-1} x_i\right)^{a_m-1}}{\left[A_m + \sum_{i=1}^m x_i (A_i - A_m)\right]^{\sum_{i=1}^m a_i}}. \tag{F.4}$$

The scalar one-loop integral of Eq. F.1 can be given as an example for Feynman parametrisation. With the identification $A_i = (k+q_i)^2 - m_i^2$ and if $A_i \neq A_j$, then

$$I_{\mathrm{s}} = \Gamma(n) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_n \frac{\delta(1-a_1-\cdots-a_n)}{\left[\sum_{i=1}^n x_i A_i\right]^n}. \tag{F.5}$$

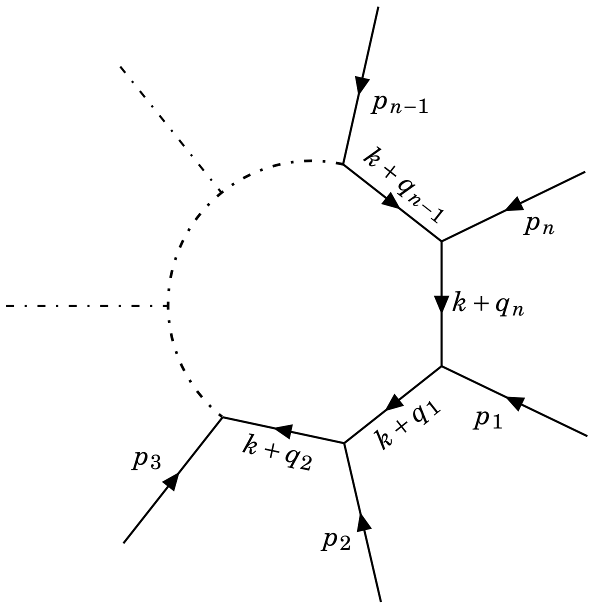


Figure F.1: **Feynman diagram for a one-loop integral**. $p_1, \cdots, p_n$ are the external momenta of the diagram while $q_i = q_{i-1} + p_i$. Because $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i$, the momentum k remains unconstrained by the momentum conservation law at the vertices.

By (i) completing the sum in the denominator to a square, $\sum_{i=1}^{m} x_i A_i = (k+K)^2 - \Delta^2$; and (ii) making a shift in the momentum k, $k \to k - K$; $I_{\rm s}$ is recasted to

$$I_{\rm s} = \Gamma(n) \int \frac{{\rm d}^d k}{(2\pi)^d} \int_0^1 {\rm d}x_1 \cdots \int_0^1 {\rm d}x_n \frac{\delta(1 - a_1 - \cdots - a_n)}{\left[k^2 + \Delta^2\right]^n}, \tag{F.6}$$

with $\Delta = \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j \left[(q_i - q_j)^2 - m_i^2 - m_j^2\right]$.

Similar operations can be applied to $I_{\rm t}^{\alpha_1 \cdots \alpha_r}$ in Eq. F.2. Unfortunately, the momentum shift $k \to k - K$ and the tensor forms in the numerator complicate the calculations. For a discussion of the tensor one-loop integral in the Feynman parametrisation see [30].

As stated above, the Feynman parametrisation introduces additional integration parameters. Nevertheless, Eq. F.6 demonstrates the power of this parametrisation. With the help of Wick rotation and dimensional regularization [17, p. 193 and p. 249], the integral over the momentum k can be preformed. The formula used is [18, p. 827]

$$\int \frac{{\rm d}^d k}{(2\pi)^d} \frac{k^{2a}}{\left[k^2 - \Delta^2\right]^b} = {\rm i}(-1)^{a-b} (4\pi)^{-d/2} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a + d/2)\Gamma(b - a - d/2)}{\Gamma(b)\Gamma(d/2)}. \tag{F.7}$$

$I_{\rm s}$ has now the form

$$I_{\rm s} = {\rm i}(-1)^n (4\pi)^{-d/2} \Gamma(n - d/2) \int_0^1 {\rm d}x_1 \cdots \int_0^1 {\rm d}x_n \delta(1 - a_1 - \cdots - a_n) \Delta^{-n+d/2}. \tag{F.8}$$

Due to the algorithm in Appendix F.2, which reduces tensor one-loop integrals to scalar one-lop integrals, Eq. F.8 was used in the calculation associated to the master thesis. The simplicity of Eq. F.8 proved to be of great help in taking different limits of the dynamical variables. However, Feynman parametrisation of tensor one-loop integrals was utilized to cross-check the algorithm in Appendix F.2 in a special limit of the dynamical variables.

F.2 Algorithm for Decomposing Tensor-Integrals into Scalar-Integrals

The matrix element $\mathcal{M}^{\mu\nu}$ of the FVVCS diagrams could be calculated by employing the reduction of tensor one-loop integrals to scalar one-loop integrals. This reduction can be done in general by considering only the tesorial character of the final result [31; 32]. However, in this paper the gauge invariance of $\mathcal{M}^{\mu\nu}$ was used to do the reduction of tensor one-loop integrals to scalar one-loop integrals.

The gauge invariance of the photon dictates that the matrix element $\mathcal{M}^{\mu\nu}$ of FVVCS can be expressed as:

$$\mathrm{i}\mathcal{M}^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2(\nu, Q^2), \tag{F.9}$$

where $\nu = \frac{p\cdot q}{M}$, $q^2 = -Q^2$, $p^2 = M^2$. $T_1(\nu, Q^2)$ and $T_2(\nu, Q^2)$ are scalar functions, which can be determined by solving the system of linear equations obtained by contracting Eq. F.9 with $g_{\mu\nu}$ and $p_\mu p_\nu$ respectively:

$$\begin{aligned} \mathrm{i}g_{\mu\nu}\mathcal{M}^{\mu\nu} &= (1-d)T_1(\nu, Q^2) + \left(1 + \frac{\nu^2}{Q^2}\right) T_2(\nu, Q^2), \\ \mathrm{i}p_\mu p_\nu \mathcal{M}^{\mu\nu} &= -M^2\left(1 + \frac{\nu^2}{Q^2}\right) T_1(\nu, Q^2) + M^2\left(1 + \frac{\nu^2}{Q^2}\right)^2 T_2(\nu, Q^2). \end{aligned} \tag{F.10}$$

Thus, if $g_{\mu\nu}\mathcal{M}^{\mu\nu}$ and $p_\mu p_\nu \mathcal{M}^{\mu\nu}$ can be expressed as a sum of scalar one-loop integrals, then also $T_1(\nu, Q^2)$ and $T_2(\nu, Q^2)$ can be written as a sum of scalar one-loop integrals, respectively.

To reduce the tensor integrals of the contractions $g_{\mu\nu}\mathcal{M}^{\mu\nu}$ and $p_\mu p_\nu \mathcal{M}^{\mu\nu}$ to scalar integrals, the following procedure was used:

1. Calculate in d-dimension the traces in the numerator. This yields multinomials in the variables k^2, $k\cdot p$, and $k\cdot q$,

2. Expand into partial fractions. This can be done by firstly replacing k^2, $k\cdot p$, and $k\cdot q$ in the numerator with suitable linear combinations of the terms in the denominator. Next, expand into a sum of fractions and make simplifications. For example

$$\begin{aligned} &\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k\cdot q}{\left[k^2 - m^2\right]^2 \left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]} \\ &\quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\frac{1}{2}\left\{-\left[k^2 - m^2\right] + \left[(k+q)^2 - m^2\right] - q^2\right\}}{\left[k^2 - m^2\right]^2 \left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]} \\ &\quad = -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2 - m^2\right]\left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]} \\ &\qquad + \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2 - m^2\right]^2 \left[(k-p)^2 - m^2\right]} \\ &\qquad - \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{q^2}{\left[k^2 - m^2\right]^2 \left[(k+q)^2 - m^2\right]\left[(k-p)^2 - m^2\right]} \end{aligned}$$

   with the replacement $k\cdot q = \frac{1}{2}\left\{-\left[k^2 - m^2\right] + \left[(k+q)^2 - m^2\right] - q^2\right\}$,

3. When no linear dependency between the numerator and denominator can be found, use the tensor like property of the integration result.

   For example, the integral

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k\cdot p)^2}{\left[k^2 - m^2\right]^2 \left[(k+q)^2 - m^2\right]} = p_\alpha p_\beta \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\alpha k^\beta}{\left[k^2 - m^2\right]^2 \left[(k+q)^2 - m^2\right]}$$

can be scalarized by finding the scalar integrals $I_1(q\cdot q)$ and $I_2(q\cdot q)$ of

$$\mathcal{A}^{\alpha\beta} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\alpha k^\beta}{\left[k^2-m^2\right]^2 \left[(k+q)^2-m^2\right]} = g^{\alpha\beta} I_1(q\cdot q) + q^\alpha q^\beta I_2(q\cdot q).$$

To find $I_1(q\cdot q)$ and $I_2(q\cdot q)$, the above mentioned can be used: (i) contract $\mathcal{A}^{\alpha\beta}$ with $g_{\alpha\beta}$ and $q_\alpha q_\beta$ to obtain a system of two linear equations, (ii) expand the $g_{\alpha\beta}\mathcal{A}^{\alpha\beta}$ and $q_\alpha q_\beta \mathcal{A}^{\alpha\beta}$ like into partial fractions, like in Step 2, (iii) solve for $I_1(q\cdot q)$ and $I_2(q\cdot q)$,

4. Use

$$\frac{1}{2p\cdot q}\left\{-\left[k^2-m^2\right]+\left[(k+q)^2-m^2\right]+\left[(k-p)^2-m^2\right]-\left[(k+q-p)^2-m^2\right]\right\}=1,$$

to simplify

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]\left[(k+q)^2-m^2\right]\left[(k-p)^2-m^2\right]\left[(k+q-p)^2-m^2\right]}$$

and bring it to a sum of three-point integrals.

If $p\cdot q=0$, such that

$$-\left[k^2-m^2\right]+\left[(k+q)^2-m^2\right]+\left[(k-p)^2-m^2\right]-\left[(k+q-p)^2-m^2\right]=0,$$

then use

$$1=\frac{\left[k^2-m^2\right]}{\left[k^2-m^2\right]}=\frac{\left[(k+q)^2-m^2\right]+\left[(k-p)^2-m^2\right]-\left[(k+q-p)^2-m^2\right]}{\left[k^2-m^2\right]}$$

to simplify

$$\begin{aligned}&\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]\left[(k+q)^2-m^2\right]\left[(k-p)^2-m^2\right]\left[(k+q-p)^2-m^2\right]}\\ &\qquad=\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\left[k^2-m^2\right]}{\left[k^2-m^2\right]} \frac{1}{\left[k^2-m^2\right]\left[(k+q)^2-m^2\right]\left[(k-p)^2-m^2\right]\left[(k+q-p)^2-m^2\right]}\end{aligned}$$

and bring it to a sum of three-point integrals,

5. Identify all the integrals that are equivalent modulo Feynman parametrisation and express the result in terms of equivalence classes/integrals.

F.3 Examples of FVVCS Matrix Elements as Functions of Scalar-Integrals

As discussed in Appendix D.2, the matrix element of FVVCS has the form

$$\mathcal{M}^{\mu\nu} = \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right) T_1(\nu,Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) T_2(\nu,Q^2).$$

By using the algorithm in Appendix F.2, the scalar functions $T_1(\nu,Q^2)$ and $T_2(\nu,Q^2)$ can be reduced to the following sums of scalar one-loop integrals

$$\begin{aligned}T_1(\nu,Q^2) = {}& b_{11}B(0)+b_{12}B\left(q^2\right)+b_{13}B\left(p^2\right)+b_{14}B\left((q-p)^2\right)+b_{15}B\left((q+p)^2\right)\\ &+c_{11}C\left(0,p^2,p^2\right)+c_{12}C\left(0,q^2,q^2\right)+c_{13}C\left(q^2,(q+p)^2,p^2\right)+c_{14}C\left(q^2,(q-p)^2,p^2\right)\\ &+d_{11}D\left(0,q^2,p^2,q^2,p^2,(q+p)^2\right)+d_{12}D\left(0,q^2,p^2,q^2,p^2,(q-p)^2\right),\end{aligned}$$

$$
\begin{aligned}
T_2(\nu,Q^2) = {} & b_{21}B(0) + b_{22}B\left(q^2\right) + b_{23}B\left(p^2\right) + b_{24}B\left((q-p)^2\right) + b_{25}B\left((q+p)^2\right) \\
& + c_{21}C\left(0,p^2,p^2\right) + c_{22}C\left(0,q^2,q^2\right) + c_{23}C\left(q^2,(q+p)^2,p^2\right) + c_{24}C\left(q^2,(q-p)^2,p^2\right) \\
& + d_{21}D\left(0,q^2,p^2,q^2,p^2,(q+p)^2\right) + d_{22}D\left(0,q^2,p^2,q^2,p^2,(q-p)^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
B\left(p_1^2\right) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]\left[(k+p_1)^2-m^2\right]}, \\
C\left(p_1^2,p_2^2,(p_1+p_2)^2\right) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]\left[(k+p_1)^2-m^2\right]\left[(k+p_1+p_2)^2-m^2\right]}, \\
D\left(p_1^2,p_2^2,p_3^2,p_4^2,(p_1+p_2)^2,(p_2+p_3)^2\right) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]\left[(k+p_1)^2-m^2\right]} \\
&\quad \times \frac{1}{\left[(k+p_1+p_2)^2-m^2\right]\left[(k+p_1+p_2+p_3)^2-m^2\right]}.
\end{aligned}
$$

The above definitions of the scalar one-loop integrals $B(\star)$, $C(\star)$, and $D(\star)$ coincide with the definitions used in the user manual of LoopTools [33] for the two-, three-, and four-point functions, respectively. The $b_\star$, $c_\star$, and $d_\star$ coefficients are given below. Note that $q^2=-Q^2$, $p^2=M^2$, $p\cdot q = M\nu$, and $d=4-2\epsilon$. The infinitesimal ϵ in the expression of the coefficients $b_\star$ is needed to assure electro-magnetic gauge invariance of the matrix element $\mathscr{M}^{\mu\nu}$ when the entire expression is expanded to zeroth order in ϵ around 0.

The following interaction Lagrangians are in one-to-one correspondence with the complete Lagrangians in Table 1.1. The interaction Lagrangian of the photon field A_μ is implied as being of QED or scalar QED type, context dependent. ϕ is a scalar field, B_μ is a vector field.

1. $\mathscr{L}_{1,\mathrm{int}} = g\bar{\psi}\psi\phi$:

$$
\begin{aligned}
b_{11} &= \frac{8}{9}(\epsilon-3) \\
b_{12} &= \frac{8}{9}(2\epsilon+3) \\
b_{13} &= b_{14} = b_{15} = 0 \\
c_{11} &= -2\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(2\nu^2+Q^2\right) \\
c_{12} &= 2\frac{1}{\nu^2+Q^2}Q^2\left(4m^2-M^2\right) + \frac{8}{3}\left(2m^2-Q^2\right) \\
c_{13} &= 2\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(2\nu^2+Q^2\right) \\
&\quad + \frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[-16m^4\left(\nu^2+Q^2\right) + 4m^2\left(M^2\left(3\nu^2+2Q^2\right)+Q^4\right)\right. \\
&\qquad \left. - M^2\left(2\nu^2+Q^2\right)\left(2\nu^2+M^2+Q^2\right)\right] \\
c_{14} &= 2\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(2\nu^2+Q^2\right) \\
&\quad - \frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[-16m^4\left(\nu^2+Q^2\right) + 4m^2\left(M^2\left(3\nu^2+2Q^2\right)+Q^4\right)\right. \\
&\qquad \left. - M^2\left(2\nu^2+Q^2\right)\left(2\nu^2+M^2+Q^2\right)\right] \\
d_{11} &= -2\frac{1}{\nu^2+Q^2}\nu M Q^2\left(4m^2-M^2\right) \\
&\quad + \frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right) - Q^2\left(2\nu^2+M^2+Q^2\right)\right) \\
d_{12} &= 2\frac{1}{\nu^2+Q^2}\nu M Q^2\left(4m^2-M^2\right) \\
&\quad + \frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right) - Q^2\left(2\nu^2+M^2+Q^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
b_{21} &= b_{22} = b_{23} = b_{24} = b_{25} = 0 \\
c_{21} &= 2\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(Q^2-2\nu^2\right) \\
c_{22} &= 6\frac{1}{\left(\nu^2+Q^2\right)^2}Q^4\left(4m^2-M^2\right) \\
c_{23} &= -2\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(Q^2-2\nu^2\right) \\
&\quad + \frac{1}{\nu M\left(\nu^2+Q^2\right)^2}\Big[-16m^4Q^2\left(\nu^2+Q^2\right)-4m^2Q^2\left(\nu^2\left(4Q^2-3M^2\right)+Q^4\right) \\
&\qquad -M^2Q^2\left(4\nu^4-M^2\left(Q^2-2\nu^2\right)+3Q^4+4\nu^2Q^2\right)\Big] \\
c_{24} &= -2\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(Q^2-2\nu^2\right) \\
&\quad - \frac{1}{\nu M\left(\nu^2+Q^2\right)^2}\Big[-16m^4Q^2\left(\nu^2+Q^2\right)-4m^2Q^2\left(\nu^2\left(4Q^2-3M^2\right)+Q^4\right) \\
&\qquad -M^2Q^2\left(4\nu^4-M^2\left(Q^2-2\nu^2\right)+3Q^4+4\nu^2Q^2\right)\Big] \\
d_{21} &= -6\frac{1}{\left(\nu^2+Q^2\right)^2}\nu M Q^4\left(4m^2-M^2\right) \\
&\quad + \frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right) \\
d_{22} &= 6\frac{1}{\left(\nu^2+Q^2\right)^2}\nu M Q^4\left(4m^2-M^2\right) \\
&\quad + \frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right)
\end{aligned}
$$

2. $\mathscr{L}_{2,\text{int}} = \mathrm{i}g\bar{\psi}\psi\gamma_5\phi$:

$$
\begin{aligned}
b_{11} &= \frac{8}{9}(\epsilon-3) \\
b_{12} &= \frac{8}{9}(2\epsilon+3) \\
b_{13} &= b_{14} = b_{15} = 0 \\
c_{11} &= 2\frac{1}{\nu^2+Q^2}M^2\left(2\nu^2+Q^2\right) \\
c_{12} &= \frac{8}{3}\left(2m^2-Q^2\right)+2\frac{1}{\nu^2+Q^2}M^2Q^2 \\
c_{13} &= -2\frac{1}{\nu^2+Q^2}M^2\left(2\nu^2+Q^2\right) \\
&\quad + \frac{1}{M\nu\left(\nu^2+Q^2\right)}M^2\left[4m^2\left(\nu^2+Q^2\right)-\left(2\nu^2+Q^2\right)\left(2\nu^2+M^2+Q^2\right)\right] \\
c_{14} &= -2\frac{1}{\nu^2+Q^2}M^2\left(2\nu^2+Q^2\right) \\
&\quad - \frac{1}{M\nu\left(\nu^2+Q^2\right)}M^2\left[4m^2\left(\nu^2+Q^2\right)-\left(2\nu^2+Q^2\right)\left(2\nu^2+M^2+Q^2\right)\right] \\
d_{11} &= 2\frac{1}{\nu^2+Q^2}\nu M^3Q^2-\frac{1}{\nu^2+Q^2}M^2\left[4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right] \\
d_{12} &= -2\frac{1}{\nu^2+Q^2}\nu M^3Q^2-\frac{1}{\nu^2+Q^2}M^2\left[4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right] \\
b_{21} &= b_{22} = b_{23} = b_{24} = b_{25} = 0 \\
c_{21} &= -2\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^2\left(Q^2-2\nu^2\right) \\
c_{22} &= -6\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^4
\end{aligned}
$$

$$c_{23} = 2\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^2\left(Q^2-2\nu^2\right)$$
$$-\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}M^2Q^2\left[4\nu^4+2\nu^2\left(-2m^2+M^2+2Q^2\right)-Q^2\left(4m^2+M^2-3Q^2\right)\right]$$

$$c_{24} = 2\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^2\left(Q^2-2\nu^2\right)$$
$$+\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}M^2Q^2\left[4\nu^4+2\nu^2\left(-2m^2+M^2+2Q^2\right)-Q^2\left(4m^2+M^2-3Q^2\right)\right]$$

$$d_{21} = 6\frac{1}{\left(\nu^2+Q^2\right)^2}\nu M^3Q^4$$
$$-\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^2\left[4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+3M^2-Q^2\right)\right]$$

$$d_{21} = -6\frac{1}{\left(\nu^2+Q^2\right)^2}\nu M^3Q^4$$
$$-\frac{1}{\left(\nu^2+Q^2\right)^2}M^2Q^2\left[4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+3M^2-Q^2\right)\right]$$

3. $\mathscr{L}_{4,\text{int}} = g\bar{\psi}\gamma_\mu\psi B_\mu$:

$$b_{11} = \frac{16}{27}(4\epsilon-3)$$
$$b_{12} = -\frac{16}{27}(\epsilon-3)+\frac{4}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}Q^4(\epsilon+1)$$
$$b_{13} = \frac{8}{3}-\frac{4}{3}\frac{1}{\nu^2+Q^2}Q^2(\epsilon+1)$$
$$b_{14} = -\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)}\nu Q^2(\epsilon+1)-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}\left(M^2\left(2\nu^2+Q^2(1-\epsilon)\right)+Q^4(\epsilon+1)\right)$$
$$b_{15} = \frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)}\nu Q^2(\epsilon+1)-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}\left(M^2\left(2\nu^2+Q^2(1-\epsilon)\right)+Q^4(\epsilon+1)\right)$$
$$c_{11} = \frac{4}{3}\frac{1}{\nu^2+Q^2}\left(2m^2+M^2\right)\left(2\nu^2+Q^2\right)$$
$$c_{12} = \frac{4}{9}\left(8m^2-4Q^2\right)-\frac{4}{3}\frac{1}{\nu^2+Q^2}Q^2\left(2m^2+M^2\right)$$
$$c_{13} = -\frac{4}{3}\frac{1}{\nu^2+Q^2}\left(2\nu^2+Q^2\right)\left(2m^2+M^2-Q^2\right)$$
$$+\frac{1}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[16m^4\left(\nu^2+Q^2\right)+4m^2Q^2\left(2\nu^2+M^2+Q^2\right)-4\nu^2\left(M^4+Q^4\right)-8\nu^4M^2-2\left(Q^3-M^2Q\right)^2\right]$$
$$c_{14} = -\frac{4}{3}\frac{1}{\nu^2+Q^2}\left(2\nu^2+Q^2\right)\left(2m^2+M^2-Q^2\right)$$
$$-\frac{1}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[16m^4\left(\nu^2+Q^2\right)+4m^2Q^2\left(2\nu^2+M^2+Q^2\right)-4\nu^2\left(M^4+Q^4\right)-8\nu^4M^2-2\left(Q^3-M^2Q\right)^2\right]$$
$$d_{11} = \frac{4}{3}\frac{1}{\nu^2+Q^2}\nu MQ^2\left(2m^2+M^2\right)$$
$$-\frac{2}{3}\frac{1}{\nu^2+Q^2}\left(2m^2+M^2\right)\left(4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right)$$
$$d_{12} = -\frac{4}{3}\frac{1}{\nu^2+Q^2}\nu MQ^2\left(2m^2+M^2\right)$$
$$-\frac{2}{3}\frac{1}{\nu^2+Q^2}\left(2m^2+M^2\right)\left(4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right)$$
$$b_{21} = 0$$

$$b_{22}=\frac{4}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}(\epsilon+3)Q^6$$

$$b_{23}=-\frac{4}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}\left(Q^4(\epsilon+1)-2\nu^2Q^2\right)$$

$$b_{24}=-\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)^2}\nu Q^4(\epsilon+3)$$
$$-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}Q^2\left[Q^4(\epsilon+3)-M^2\left(Q^2(\epsilon+1)-2\nu^2\right)\right]$$

$$b_{25}=\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)^2}\nu Q^4(\epsilon+3)$$
$$-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}Q^2\left[Q^4(\epsilon+3)-M^2\left(Q^2(\epsilon+1)-2\nu^2\right)\right]$$

$$c_{21}=-\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(2m^2+M^2\right)\left(Q^2-2\nu^2\right)$$

$$c_{22}=-4\frac{1}{\left(\nu^2+Q^2\right)^2}Q^4\left(2m^2+M^2\right)$$

$$c_{23}=\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(Q^2-2\nu^2\right)\left(2m^2+M^2-Q^2\right)$$
$$+\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}Q^2\left[8m^4\left(\nu^2+Q^2\right)+6m^2Q^2\left(2\nu^2+M^2+Q^2\right)\right.$$
$$\left.-2\nu^2\left(M^4+Q^4\right)-4\nu^4M^2+\left(Q^3-M^2Q\right)^2\right]$$

$$c_{24}=\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(Q^2-2\nu^2\right)\left(2m^2+M^2-Q^2\right)$$
$$-\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}Q^2\left[8m^4\left(\nu^2+Q^2\right)+6m^2Q^2\left(2\nu^2+M^2+Q^2\right)\right.$$
$$\left.-2\nu^2\left(M^4+Q^4\right)-4\nu^4M^2+\left(Q^3-M^2Q\right)^2\right]$$

$$d_{21}=\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}\nu MQ^4\left(2m^2+M^2\right)$$
$$-\frac{2}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(2m^2+M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right)$$

$$d_{22}=-\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}\nu MQ^4\left(2m^2+M^2\right)$$
$$-\frac{2}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(2m^2+M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right)$$

4. $\mathscr{L}_{5,\text{int}}=g\bar{\psi}\gamma_\mu\gamma_5\psi B_\mu$:

$$b_{11}=\frac{16}{27}(4\epsilon-3)$$

$$b_{12}=-\frac{16}{27}(\epsilon-3)+\frac{4}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}Q^4(\epsilon+1)$$

$$b_{13}=\frac{8}{3}-\frac{4}{3}\frac{1}{\nu^2+Q^2}Q^2(\epsilon+1)$$

$$b_{14}=-\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)}\nu Q^2(\epsilon+1)-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}\left(M^2\left(2\nu^2-Q^2(\epsilon-1)\right)+Q^4(\epsilon+1)\right)$$

$$b_{15}=\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)}\nu Q^2(\epsilon+1)-\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)}\left(M^2\left(2\nu^2-Q^2(\epsilon-1)\right)+Q^4(\epsilon+1)\right)$$

$$c_{11}=-\frac{4}{3}\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(2\nu^2+Q^2\right)$$

$$c_{12}=\frac{16}{9}\left(2m^2-Q^2\right)+\frac{4}{3}\frac{1}{\nu^2+Q^2}Q^2\left(4m^2-M^2\right)$$

$$c_{13} = \frac{4}{3}\frac{1}{\nu^2+Q^2}\left(2\nu^2+Q^2\right)\left(4m^2-M^2+Q^2\right)$$
$$-\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[4\nu^4\left(2m^2+M^2\right)+Q^2\left(4m^2-M^2+Q^2\right)^2 + 2\nu^2\left(8m^4+m^2\left(10Q^2-6M^2\right)+M^4+Q^4\right)\right]$$

$$c_{23} = \frac{4}{3}\frac{1}{\nu^2+Q^2}\left(2\nu^2+Q^2\right)\left(4m^2-M^2+Q^2\right)$$
$$+\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)}\left[4\nu^4\left(2m^2+M^2\right)+Q^2\left(4m^2-M^2+Q^2\right)^2 + 2\nu^2\left(8m^4+m^2\left(10Q^2-6M^2\right)+M^4+Q^4\right)\right]$$

$$d_{11} = -\frac{4}{3}\frac{1}{\nu^2+Q^2}\nu MQ^2\left(4m^2-M^2\right)$$
$$+\frac{2}{3}\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right)$$

$$d_{12} = \frac{4}{3}\frac{1}{\nu^2+Q^2}\nu MQ^2\left(4m^2-M^2\right)$$
$$+\frac{2}{3}\frac{1}{\nu^2+Q^2}\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)-Q^2\left(2\nu^2+M^2+Q^2\right)\right)$$

$$b_{21} = 0$$

$$b_{22} = \frac{4}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}Q^6(\epsilon+3)$$

$$b_{23} = \frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(2\nu^2-Q^2(\epsilon+1)\right)$$

$$b_{24} = -\frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)^2}\nu Q^4(\epsilon+3)+\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}Q^2\left(M^2\left(Q^2(\epsilon+1)-\nu^2\right)-2Q^4(\epsilon+3)\right)$$

$$b_{25} = \frac{4}{3}\frac{1}{M\left(\nu^2+Q^2\right)^2}\nu Q^4(\epsilon+3)+\frac{2}{3}\frac{1}{M^2\left(\nu^2+Q^2\right)^2}Q^2\left(M^2\left(Q^2(\epsilon+1)-\nu^2\right)-2Q^4(\epsilon+3)\right)$$

$$c_{21} = \frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(Q^2-2\nu^2\right)$$

$$c_{22} = 4\frac{1}{\left(\nu^2+Q^2\right)^2}Q^4\left(4m^2-M^2\right)$$

$$c_{23} = -\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(Q^2-2\nu^2\right)\left(4m^2-M^2+Q^2\right)$$
$$-\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}Q^2\left[16m^4\left(\nu^2+Q^2\right)+4m^2\left(2\nu^4+\nu^2\left(7Q^2-3M^2\right)+2Q^4\right) + 2\nu^2\left(M^4+Q^4\right)+4\nu^4M^2+\left(Q^3-M^2Q\right)^2\right]$$

$$c_{24} = -\frac{4}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(Q^2-2\nu^2\right)\left(4m^2-M^2+Q^2\right)$$
$$+\frac{2}{3}\frac{1}{\nu M\left(\nu^2+Q^2\right)^2}Q^2\left[16m^4\left(\nu^2+Q^2\right)+4m^2\left(2\nu^4+\nu^2\left(7Q^2-3M^2\right)+2Q^4\right) + 2\nu^2\left(M^4+Q^4\right)+4\nu^4M^2+\left(Q^3-M^2Q\right)^2\right]$$

$$d_{21} = -4\frac{1}{\left(\nu^2+Q^2\right)^2}\nu MQ^4\left(4m^2-M^2\right)$$
$$+\frac{2}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right)$$

$$d_{22} = 4\frac{1}{\left(\nu^2+Q^2\right)^2}\nu MQ^4\left(4m^2-M^2\right)$$
$$+\frac{2}{3}\frac{1}{\left(\nu^2+Q^2\right)^2}Q^2\left(4m^2-M^2\right)\left(4m^2\left(\nu^2+Q^2\right)+Q^2\left(-2\nu^2-3M^2+Q^2\right)\right)$$

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