Exercise sheet 12
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (40 points) : Lifetime of the 2p level of Hydrogen

Calculate the lifetime of the $\psi_{210}(\vec{r})$ state, in the Bohr theory for the Hydrogen atom, due to its decay (electric dipole transition) into into the $\psi_{100}(\vec{r})$ state with

$$\psi_{210}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \frac{\cos\theta}{2\sqrt{2a^3}} \frac{r}{a} e^{-\frac{r}{2a}} \tag{1}$$

and

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \frac{2}{\sqrt{a^3}} e^{-\frac{r}{a}}, \tag{2}$$

with Bohr radius $a = 1/(m\alpha)$ and fine-structure constant $\alpha = e^2/(4\pi) \simeq 1/137$ (both in natural units $\hbar = c = 1$). Also, give a numerical result for the lifetime (in seconds).

Exercise 2. (60 points) : Hyperfine splitting in hydrogen, 21 cm line

The interaction between an electron and a magnetic field is given by

$$\hat{H}_{int} = -\vec{\mu} \cdot \hat{\vec{B}}, \tag{3}$$

where μ is the magnetic moment

$$\vec{\mu} = \frac{e}{2m} \frac{\hbar}{c} \vec{\sigma} \tag{4}$$

of the electron.

a) (*10 p.*) Express $\hat{H}_{int}$ through its normal mode expansion. *Hint*: Use

$$\hat{\vec{A}} = \sum_{\vec{k},\sigma} N_k \left\{ \hat{a}_{\vec{k},\sigma} \vec{\epsilon}_{\vec{k},\sigma} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{k},\sigma} \vec{\epsilon}^{\,*}_{\vec{k},\sigma} e^{-i\vec{k}\cdot\vec{x}} \right\}, \tag{5}$$

where $\epsilon_{\vec{k},\sigma}$ is the photon polarisation vector.

b) (*50 p.*) The 1S-state with $f = 1$ has a slightly higher energy than the 1S-state with $f = 0$. In the transition between the inital state

$$|\psi_i\rangle = |1S\rangle \, |\uparrow\rangle_e \, |\uparrow\rangle_p \,, \tag{6}$$

and the final state (with total electron + proton spin equal to zero)

$$|\psi_f\rangle = |1S\rangle \, \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_e \, |\downarrow\rangle_p - |\downarrow\rangle_e \, |\uparrow\rangle_p \right\}, \tag{7}$$

a photon with $\lambda \approx 21$cm is emitted.
Derive the lifetime of this transition and give a numerical value (in seconds and in years).
Hint: Calculate the matrix element

$$\langle f| \, \hat{H}_{int} \, |i\rangle \tag{8}$$

in dipole approximation and apply Fermi's golden rule. The expected answer is:

$$\frac{1}{\tau_{i\to f}} = \frac{e^2}{4\pi} \frac{1}{3} \frac{\hbar}{m^2 c^2} k_0^3, \tag{9}$$

with $k_0 = \frac{2\pi}{\lambda}$.