# Exercise sheet 11
# Theoretical Physics 5 : WS 2021/2022
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## Exercise 1. (40 points) : Scalar Quantum Electrodynamics

Consider the Lagrangian for the charged Klein-Gordon Field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi \tag{1}$$

- Check that this Lagrangian is not invariant under U(1) gauge transformation of the form $\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x)$;
- Replace the derivatives by the convariant one $D_\mu = \partial_\mu + ieA_\mu$ and check whether now the Lagrangian is invariant under the local symmetry;
- Write down the interaction Lagrangian and identify the terms.

## Exercise 2. (20 points) : The Hamiltonian of the electromagnetic field

Using the normal mode expansion of the photon field

$$A^\mu(\vec{x}, t) \;=\; \sum_{\vec{k}} \sum_{\lambda=0}^{3} \left( \frac{\hbar\, c^2}{2\,\omega_k\, L^3} \right)^{1/2} \left[ a(\vec{k})\, \epsilon^\mu(\vec{k}, \lambda)\, e^{-i\,k\cdot x} + a^\dagger(\vec{k})\, \epsilon^{\mu *}(\vec{k}, \lambda)\, e^{i\,k\cdot x} \right]$$

and the commutator relations

$$\begin{aligned}
&[a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')] = \xi_\lambda\, \delta_{\lambda,\lambda'}\, \delta_{\vec{k},\vec{k}'}, \quad \xi_0 = -1,\ \xi_{i=1,2,3} = +1 \\
&[a(\vec{k}, \lambda), a(\vec{k}', \lambda')] = 0 \\
&[a^\dagger(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')] = 0
\end{aligned}$$

show that the Hamiltonian $H = \int d^3x N\left( (\Pi^\nu(x) \dot{A}_\nu(x) - \mathcal{L}(x)) \right)$, $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ takes the following form

$$H = \sum_{\vec{k}} \hbar\, \omega_k \left( -a^\dagger(\vec{k}, 0)\, a(\vec{k}, 0) + \sum_{i=1}^{3} a^\dagger(\vec{k}, i)\, a(\vec{k}, i) \right) \tag{2}$$

## Exercise 3. (40 points) : Gauge transformation

Let $|\Psi_T\rangle$ be a state which contains only transverse photons. Show that replacing $|\Psi_T\rangle$ by $|\Psi'_T\rangle$

$$|\Psi'_T\rangle = \left\{1 + \alpha \left[a^\dagger(\vec{k}, 3) - a^\dagger(\vec{k}, 0)\right]\right\} |\Psi_T\rangle \,,$$
$$\alpha = \text{constant}$$

corresponds to a gauge transformation:

$$\langle\Psi'_T|A^\mu(x)|\Psi'_T\rangle = \langle\Psi_T|A^\mu(x) + \partial^\mu\Lambda|\Psi_T\rangle \,,$$

where $\Lambda$ is given by

$$\Lambda(x) = \left(\frac{2\hbar\, c^2}{\omega_k^3\, L^3}\right)^{1/2} \Re\left(i\alpha e^{-ik\cdot x}\right) .$$