Exercise sheet 10
Theoretical Physics 5 : WS 2021/2022
Lecturers : Prof. M. Vanderhaeghen, Dr. I. Danilkin

03.01.2022

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (30 points)
Ground state Dirac Coulomb wave functions

Consider the $1s_{1/2}$ ground state of an electron in a hydrogen-like ion with nuclear charge Z. Derive the **normalized** radial Dirac Coulomb wave functions $F(\rho)$ and $G(\rho)$ from the power series expansion method considered at the lecture.
Hints: Recall

$$F(\rho) = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{n'} a_m \rho^{m+\gamma}$$

$$G(\rho) = \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{n'} b_m \rho^{m+\gamma},$$

where $n' = n - (j + 1/2)$, $\rho = 2\sqrt{k_1 k_2} r$ and $k_{1,2} = \frac{1}{\hbar c}(\pm E + m_0 c^2)$. As was shown $\gamma = \sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2}$ and the spectrum is

$$E_{nj} = \frac{m_0 c^2}{\sqrt{1 + \left(\frac{Z\alpha}{n-(j+1/2)+\gamma}\right)^2}}.$$

The $1s_{1/2}$ state corresponds to $n = 1$ and $j = 1/2$ so that F and G consist of a single term. The normalization condition implies

$$\int_0^\infty \mathrm{d}r \, \left[F^2(r) + G^2(r)\right] = 1.$$

Use that the gamma function is defined via

$$\Gamma(z) = \int_0^\infty \mathrm{d}x \, x^{z-1} e^{-x}.$$

Exercise 2. (45 points) : Dirac field

Using the normal mode expansion of the Dirac field

$$\psi(\vec{x},t) = \sum_{\vec{p}}\sum_{s_z}\left(\frac{m_0c^2}{E_pV}\right)^{1/2}\left[b(\vec{p},s_z)\,u(\vec{p},s_z)e^{-\frac{i}{\hbar}p\cdot x} + d^\dagger(\vec{p},s_z)\,v(\vec{p},s_z)e^{\frac{i}{\hbar}p\cdot x}\right]$$

$$\bar{\psi}(\vec{x},t) = \sum_{\vec{p}}\sum_{s_z}\left(\frac{m_0c^2}{E_pV}\right)^{1/2}\left[b^\dagger(\vec{p},s_z)\,\bar{u}(\vec{p},s_z)e^{\frac{i}{\hbar}p\cdot x} + d(\vec{p},s_z)\,\bar{v}(\vec{p},s_z)e^{-\frac{i}{\hbar}p\cdot x}\right]$$

and the equal-time creation and annihilation operators anticommutation relations

$$\{b(\vec{p},s_z),b^\dagger(\vec{p}',s_z')\} = \delta_{\vec{p},\vec{p}'}\delta_{s_z,s_z'},$$
$$\{d(\vec{p},s_z),d^\dagger(\vec{p}',s_z')\} = \delta_{\vec{p},\vec{p}'}\delta_{s_z,s_z'},$$
$$\{b(\vec{p},s_z),b(\vec{p}',s_z')\} = 0, \quad \{d(\vec{p},s_z),d(\vec{p}',s_z')\} = 0,$$
$$\{b(\vec{p},s_z),d(\vec{p}',s_z')\} = 0, \quad \{d(\vec{p},s_z),b(\vec{p}',s_z')\} = 0,$$
$$\{b(\vec{p},s_z),d^\dagger(\vec{p}',s_z')\} = 0, \quad \{d(\vec{p},s_z),b^\dagger(\vec{p}',s_z')\} = 0;$$

a) (*10 p.*) Show that the following equal time anticommutation relations hold:

$$\{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{x}',t)\} = 0,$$
$$\left\{\psi_\alpha^\dagger(\vec{x},t),\psi_\beta^\dagger(\vec{x}',t)\right\} = 0;$$

b) (*20 p.*) Express $H = c\int \mathrm{d}^3\vec{x}\,N\left(\bar{\psi}(-i\hbar\gamma^i\partial_i + m_0c)\psi\right)$ in terms of creation and annihilation operators.

c) (*15 p.*) Do the same for the momentum $\vec{P} = -i\hbar\int \mathrm{d}^3\vec{x}\,N\left(\psi^\dagger\vec{\nabla}\psi\right)$.

Exercise 3. (25 points)

Calculate $[H, b^\dagger(\vec{p},s_z)b(\vec{p},s_z)]$.