

IV

RELATIVISTIC
FERMIONS

- IV 1) DIRAC EQUATION
- IV 2) LORENTZ COVARIANCE OF DIRAC EQUATION
- IV 3) DIRAC PARTICLE IN CENTRAL POTENTIAL
- IV 4) QUANTIZATION OF DIRAC FIELD

1) DIRAC EQUATION

• DERIVATION OF DIRAC EQ.

↳ SCHRÖDINGER EQ.

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

↳ DIRAC (1928) WAS LOOKING FOR A RELATIVISTIC COVARIANT VERSION OF SCHRÖDINGER EQ. LINEAR IN SPACE-TIME DERIVATIVES

$$\left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right] \Psi = i\hbar \frac{\partial \Psi}{\partial t} = c \hat{\alpha} \cdot \hat{p} + \hat{\beta} m_0 c^2$$

WITH $\hat{p} = -i\hbar \vec{\nabla}$

$\hat{\alpha}_i$ CANNOT BE NUMBERS AS EQ. SHOULD BE INVARIANT UNDER SPATIAL ROTATIONS



TRY: $\hat{\alpha}_i$ ARE $N \times N$ MATRICES, Ψ : $N \times 1$ COLUMN MATRIX

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

$$\left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau} \psi_\tau$$

$$= i\hbar \frac{\partial \psi_\sigma}{\partial t}$$

SUM OVER $\tau = 1 \dots N$ (SUMMATION CONVENTION)

NOTE : σ, τ ARE USUAL MATRIX INDICES!
WE DO NOT NEED TO MAKE DISTINCTION
BETWEEN COVARIANT / CONTRAVARIANT
AS IN CASE OF FOUR-VECTORS

↳ WE LIKE TO RECOVER RELATIVISTIC ENERGY-MOMENTUM
RELATION FOR FREE PARTICLE

⇓
POSSIBLE IF EACH COMPONENT ψ_σ SATISFIES
KLEIN-GORDON EQUATION

↪ REQUIRE $\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2} \right) \psi_\sigma = 0$

PROOF

$$i\hbar \frac{\partial \psi_\sigma}{\partial t} = \left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau} \psi_\tau$$

⇓ $i\hbar \frac{\partial}{\partial t}$

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right]_{\sigma\tau} i\hbar \frac{\partial \psi_\tau}{\partial t}$$

N x N MATRIX EQ.

$$\begin{aligned}
 -\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} &= \left[-i\hbar c \sum_{i=1}^3 \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0 c^2 \right] \\
 &\cdot \left[-i\hbar c \sum_{j=1}^3 \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0 c^2 \right] \Psi \\
 &= -\hbar^2 c^2 \sum_{i,j=1}^3 \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \Psi}{\partial x^i \partial x^j} \\
 &\quad - i\hbar m_0 c^3 \sum_{i=1}^3 (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \Psi}{\partial x^i} \\
 &\quad + (m_0 c^2)^2 \hat{\beta}^2 \Psi
 \end{aligned}$$



WE WANT THIS TO BE IN FORM

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi + \frac{m_0^2 c^2}{\hbar^2} \Psi = 0$$



REQUIRE:

$$\begin{aligned}
 \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2 \delta_{ij} \mathbb{I}_{N \times N} \\
 \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0 \\
 \hat{\alpha}_i^2 = \hat{\beta}^2 &= \mathbb{I}_{N \times N}
 \end{aligned}$$

↳ IN ADDITION : WE WANT \hat{H} TO BE HERMITIAN

↓
 $\hat{\alpha}_i, \hat{\beta}$ HAVE TO BE HERMITIAN

$$\begin{array}{l} \hat{\alpha}_i^\dagger = \hat{\alpha}_i \\ \hat{\beta}^\dagger = \hat{\beta} \end{array}$$

(REAL EIGENVALUES)

• $\hat{\alpha}_i^2 = \hat{\beta}^2 = 1 \Rightarrow$ EIGENVALUES ARE ± 1

• $\hat{\alpha}_i = -\hat{\beta} \hat{\alpha}_i \hat{\beta}$

↓

$$\text{Tr } \hat{\alpha}_i = -\text{Tr} \{ \hat{\beta} \hat{\alpha}_i \hat{\beta} \} = -\text{Tr} \{ \hat{\alpha}_i \}$$

↓

$$\text{Tr } \hat{\alpha}_i = 0$$

ANALOGOUSLY $\text{Tr } \hat{\beta} = 0$ } TRACELESS.

• FOR $N=2 \Rightarrow$ ONLY 3 ANTI-COMMUTING MATRICES POSSIBLE (PAULI-MATRICES)

• SINCE $\hat{\alpha}_i, \hat{\beta}$ HAVE EIGENVALUES ± 1 & ARE TRACELESS THEY MUST HAVE SAME NUMBER OF $+1$ AS -1 EIGENVALUES \Rightarrow ONLY EVEN N ARE POSSIBLE

↳ SMALLEST POSSIBLE DIMENSION: N = 4

$\hat{\alpha}_i, \hat{\beta}$ DIRAC MATRICES : (4 x 4 MATRICES)



DIRAC REPRESENTATION

$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}$	$\hat{\beta}_i = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}$
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WITH $\hat{\sigma}_i$ PAULI-MATRICES (2 x 2 MATRICES)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

NOTE $\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2 \delta_{ij} \mathbb{1}_{2 \times 2}$

↳ $\hat{\alpha}_i, \hat{\beta}$ TRACELESS & HERMITIAN

↳ CHECK OF ANTI-COMMUTATION RELATIONS

$$\hat{\alpha}_i^2 = \begin{pmatrix} \hat{\sigma}_i^2 & 0 \\ 0 & \hat{\sigma}_i^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{2 \times 2} & \\ & \mathbb{1}_{2 \times 2} \end{pmatrix} \stackrel{!}{=} \mathbb{1}_{4 \times 4}$$

$$\hat{\beta}^2 \stackrel{!}{=} \mathbb{1}_{4 \times 4}$$

$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} \stackrel{!}{=} 0$$

$$\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix} \stackrel{!}{=} 2 \delta_{ij} \cdot \mathbb{1}_{4 \times 4}$$

~> WE CAN CHOOSE ANOTHER REPRESENTATION OF $\hat{\alpha}_i, \hat{\beta}$ BY UNITARY TF.

e.g. $\hat{\alpha}'_i = U \hat{\alpha}_i U^{-1}$
 $\hat{\beta}' = U \hat{\beta} U^{-1}$

$$\begin{aligned} \hat{\alpha}'_i \hat{\alpha}'_j + \hat{\alpha}'_j \hat{\alpha}'_i &= U \hat{\alpha}_i U^{-1} U \hat{\alpha}_j U^{-1} + U \hat{\alpha}_j U^{-1} U \hat{\alpha}_i U^{-1} \\ &= U \left(\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i \right) U^{-1} \\ &= 2 \delta_{ij} \underbrace{\mathbb{1}_{4 \times 4}} \end{aligned}$$

⇓
 PHYSICS DOES NOT DEPEND ON SPECIFIC REPRESENTATION OF DIRAC MATRICES $\hat{\alpha}_i, \hat{\beta}$

↳ ABOVE DIRAC REPRESENTATION IS OFTEN CONVENIENT AND USED IN MANY CASES

∴ **DIRAC EQUATION** (4x1 MATRIX EQ.)

$$\left[-i\hbar c \hat{\alpha} \cdot \nabla + \hat{\beta} m_0 c^2 \right] \underline{\Psi} = i\hbar \frac{\partial \underline{\Psi}}{\partial t}$$

$\hat{\alpha} \equiv (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ EACH $\hat{\alpha}_i$: 4x4 MATRIX

• CONTINUITY EQUATION

WE LIKE TO HAVE AN EQ. FOR 'PROBABILITY DENSITY'

$$\underline{\Psi}^+ \underline{\Psi}$$

~> DIRAC EQ.

$$-i\hbar c \hat{\alpha}_i \frac{\partial \underline{\Psi}}{\partial x^i} + \hat{\beta} m_0 c^2 \underline{\Psi} = i\hbar \frac{\partial \underline{\Psi}}{\partial t} \quad (*)$$

~> HERMITION CONJUGATE EQ.

$$+i\hbar c \frac{\partial \underline{\Psi}^+}{\partial x^i} \hat{\alpha}_i + \underline{\Psi}^+ \hat{\beta} m_0 c^2 = -i\hbar \frac{\partial \underline{\Psi}^+}{\partial t} \quad (**)$$

~> $\underline{\Psi}^+ (*) - (**)\underline{\Psi}$

$$i\hbar \left(\underline{\Psi}^+ \frac{\partial \underline{\Psi}}{\partial t} + \frac{\partial \underline{\Psi}^+}{\partial t} \underline{\Psi} \right)$$

$$= -i\hbar c \left(\underline{\Psi}^+ \hat{\alpha}_i \frac{\partial \underline{\Psi}}{\partial x^i} + \frac{\partial \underline{\Psi}^+}{\partial x^i} \hat{\alpha}_i \underline{\Psi} \right)$$

$$+ m_0 c^2 \left(\underline{\Psi}^+ \hat{\beta} \underline{\Psi} - \underline{\Psi}^+ \hat{\beta} \underline{\Psi} \right)$$

⇓

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$

CONTINUITY EQ.

$\rho = \underline{\Psi}^+ \underline{\Psi}$	$\underline{J} = c \underline{\Psi}^+ \hat{\alpha} \underline{\Psi}$
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ρ : PROBABILITY DENSITY
(POSITIVE DEFINITE)

\vec{J} : PROBABILITY CURRENT $\vec{J} = (J^1, J^2, J^3)$

$$J^i \equiv c \Psi^\dagger \hat{\alpha}^i \Psi$$

• SOLUTIONS OF DIRAC EQ. FOR FREE PARTICLE

↳ $\Psi(\vec{x}, t) = \mathcal{N}(\vec{x}) e^{-\frac{i}{\hbar} Et}$

$$\left[-i\hbar c \hat{\alpha} \cdot \vec{\nabla} + \hat{\beta} m_0 c^2 \right] \mathcal{N}(\vec{x}) = E \mathcal{N}(\vec{x})$$

$\mathcal{N}(\vec{x})$ IS 4 x 1 COLUMN MATRIX
↳ SPINOR

$$\mathcal{N}(\vec{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad \text{WITH } \varphi \text{ \& \ } \chi$$

2 x 1 COLUMN MATRICES

$$\left[-i\hbar c \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{\nabla} \\ \vec{\sigma} \cdot \vec{\nabla} & 0 \end{pmatrix} + m_0 c^2 \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \right] \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = E \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

$$\begin{cases} -i\hbar c \vec{\sigma} \cdot \vec{\nabla} \chi + m_0 c^2 \varphi = E \varphi \\ -i\hbar c \vec{\sigma} \cdot \vec{\nabla} \varphi - m_0 c^2 \chi = E \chi \end{cases}$$

SOLUTIONS WITH GOOD MOMENTUM

$$\psi(\bar{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{+i\frac{1}{\hbar} \bar{p} \cdot \bar{x}}$$

WITH φ, χ INDEP. OF \bar{x}

$$\begin{cases} c \vec{\sigma} \cdot \vec{p} \chi + m_0 c^2 \varphi = E \varphi \\ c \vec{\sigma} \cdot \vec{p} \varphi - m_0 c^2 \chi = E \chi \end{cases}$$

2° EQ:
$$\chi = \frac{1}{E + m_0 c^2} c \vec{\sigma} \cdot \vec{p} \varphi$$



IN 1° EQ.
$$c \vec{\sigma} \cdot \vec{p} \chi = (E - m_0 c^2) \varphi$$



$$\frac{c^2}{E + m_0 c^2} (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) \varphi = (E - m_0 c^2) \varphi$$



$$\frac{c^2}{E + m_0 c^2} \vec{p}^2 = (E - m_0 c^2)$$

$$c^2 \vec{p}^2 = E^2 - (m_0 c^2)^2$$

∴
$$E^2 = c^2 p^2 + m_0^2 c^4$$

RELATIVISTIC ENERGY - MOMENTUM RELATION

• 'POSITIVE ENERGY' / PARTICLE SOLUTIONS

CAN BE WRITTEN IN FORM

$$\underline{\Psi}(x, t) = e^{-\frac{i}{\hbar} \bar{P} \cdot x} \underline{U}_{\kappa}(\bar{P})$$

$\kappa = 1, 2$

U_{κ} IS 4×1 SPINOR

WITH $P^{\mu} \left(\frac{E_{\bar{P}}}{c}, \bar{P} \right) \quad x^{\mu} (ct, \bar{x})$

$$E_{\bar{P}} = \sqrt{c^2 \bar{P}^2 + m_0^2 c^4}$$

$U_{\kappa}(\bar{P})$ IS SOLUTION OF

$$\left(c \bar{x} \cdot \bar{P} + \beta m_0 c^2 \right) U_{\kappa}(\bar{P}) = E_{\bar{P}} U_{\kappa}(\bar{P})$$

WITH $\left\| U_{\kappa}(\bar{P}) = \begin{pmatrix} \varphi_{\kappa} \\ \frac{c \bar{\sigma} \cdot \bar{P}}{E_{\bar{P}} + m_0 c^2} \varphi_{\kappa} \end{pmatrix} \right. \quad \kappa = 1, 2$

φ_{κ} DENOTES 2-COMPONENT SPINOR (2×1 MATRIX)

2 INDEPENDENT SOLUTIONS ($\kappa = 1, 2$)

• FOR PARTICLE AT REST ($\bar{P} = 0$)

$$\varphi_1 = N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\varphi_2 = N \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

N: NORMALIZATION FACTOR

$$U_1(\bar{0}) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_2(\bar{0}) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

↳ 'NEGATIVE ENERGY' / ANTI-PARTICLE SOLUTIONS

CAN BE WRITTEN IN FORM

$$\underline{\underline{\Psi(\vec{x}, t) = e^{+\frac{i}{\hbar} P \cdot X} \mathcal{U}_\kappa(\vec{p})}}$$

\mathcal{U}_κ IS 4x1 SPINOR

WITH $P^\mu = \left(\frac{E_{\vec{p}}}{c}, \vec{p} \right) \quad X^\mu (ct, \vec{x})$

$$E_{\vec{p}} = \sqrt{c^2 \vec{p}^2 + m_0^2 c^4}$$

$\mathcal{U}_\kappa(\vec{p})$ IS SOLUTION OF

$$\left(-c \vec{\alpha} \cdot \vec{p} + \beta m_0 c^2 \right) \mathcal{U}_\kappa(\vec{p}) = -E_{\vec{p}} \mathcal{U}_\kappa(\vec{p})$$

DENOTE $\mathcal{U}_\kappa(\vec{p}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$

$$\begin{cases} -c \vec{\sigma} \cdot \vec{p} \chi + m_0 c^2 \varphi = -E_{\vec{p}} \varphi \\ -c \vec{\sigma} \cdot \vec{p} \varphi - m_0 c^2 \chi = -E_{\vec{p}} \chi \end{cases}$$

FROM 1^o EQ.

$$\varphi = \frac{1}{E_{\vec{p}} + m_0 c^2} c \vec{\sigma} \cdot \vec{p} \chi$$

INSERT IN 2^o EQ.

$$-\frac{c^2 \vec{p}^2}{E_{\vec{p}} + m_0 c^2} \chi = -(E_{\vec{p}} - m_0 c^2) \chi$$



$$c^2 \bar{p}^2 = E_{\bar{p}}^2 - m_0^2 c^4$$

$$E_{\bar{p}}^2 = c^2 \bar{p}^2 + m_0^2 c^4$$

WE FIND AGAIN RELATIVISTIC ENERGY-MOMENTUM RELATION

$$U_{\kappa}(\bar{p}) = \begin{pmatrix} \frac{c \vec{\sigma} \cdot \bar{p}}{E_{\bar{p}} + m_0 c^2} \chi_{\kappa} \\ \chi_{\kappa} \end{pmatrix}$$

DESCRIBES ANTI-PARTICLE WITH MASS m_0 ,
MOMENTUM \bar{p} ,
ENERGY $E_{\bar{p}}$

- FOR ANTI-PARTICLE AT REST ($\bar{p} = 0$)
WE MAKE OPPOSITE CHOICE (SEE LATER)

$$\chi_1 \equiv N \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\chi_2 \equiv N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U_1(\bar{0}) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$U_2(\bar{0}) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

• HELICITY

BOTH PARTICLE SOLUTION $u_{\mu}(\vec{p})$ AND

ANTI-PARTICLE SOLUTION $v_{\mu}(\vec{p})$

HAVE 2-FOLD DEGENERACY ($\mu = 1, 2$)

↳ PARTICLE AT REST ($\vec{p} = 0$)

INTRODUCE

OPERATOR

$$\bar{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

↑
4x4 MATRIX

WITH $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

2x2 PAULI-MATR.

$$\sum_3 u_1(\vec{0}) = u_1(\vec{0})$$

"SPIN-UP" SOLUTION

$$\sum_3 u_2(\vec{0}) = -u_2(\vec{0})$$

"SPIN-DOWN" SOLUTION

SPIN-OPERATOR $\bar{S} \equiv \frac{\hbar}{2} \bar{\Sigma}$

$u_1(\vec{0})$ HAS SPIN $+\frac{\hbar}{2}$ ALONG Z-AXIS

$u_2(\vec{0})$ " " $-\frac{\hbar}{2}$ " " "

↳ ANALOGOUSLY FOR ANTI-PARTICLE AT REST

$$\sum_3 v_1(\vec{0}) = -v_1(\vec{0})$$

$$\sum_3 v_2(\vec{0}) = +v_2(\vec{0})$$

↳ FOR Z-COMPONENT OF $\hat{\Sigma}$ TO BE
A GOOD QUANTUM NUMBER, IT HAS
TO COMMUTE WITH \hat{H}

DIRAC EQ.

$$\left[-i\hbar c \vec{\alpha} \cdot \vec{\nabla} + \beta m_0 c^2 \right] \bar{\Psi} = i\hbar \frac{\partial}{\partial t} \bar{\Psi}$$

$$\downarrow \hat{p} \equiv -i\hbar \vec{\nabla}$$

$$\left[c \vec{\alpha} \cdot \hat{p} + \beta m_0 c^2 \right] \bar{\Psi} = i\hbar \frac{\partial \bar{\Psi}}{\partial t}$$

\hat{H}_D

DIRAC HAMILTONIAN

↳ $[\hat{H}_D, \Sigma_3]$

COMMUTATOR

↳ AS BOTH β & Σ_3
ARE DIAGONAL

$$= [c \vec{\alpha} \cdot \hat{p}, \Sigma_3]$$

$$= c \left\{ \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} - \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} \right\}$$

$$= c \begin{pmatrix} 0 & [\vec{\sigma} \cdot \vec{p}, \sigma_3] \\ [\vec{\sigma} \cdot \vec{p}, \sigma_3] & 0 \end{pmatrix}$$

$$\downarrow \text{USE } [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$$

$$= c 2i \epsilon_{i3k} p_i \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\hat{S}_3 = \frac{\hbar}{2} \Sigma_3$$

$$[\hat{H}_D, \Sigma_3] = i\hbar c \begin{pmatrix} 0 & (\vec{\sigma} \times \vec{p})_3 \\ (\vec{\sigma} \times \vec{p})_3 & 0 \end{pmatrix}$$

$$= i\hbar c (\vec{\sigma} \times \vec{p})_3$$

$\neq 0$ FOR $\vec{p} \neq 0$ AND \vec{p} NOT ALONG Z-AXIS

\hat{H}_D AND Σ_3 DO NOT COMMUTE FOR $\vec{p} \neq 0$ AND \vec{p} NOT ALONG Z-AXIS

ONLY FOR $\vec{p} = 0$ OR \vec{p} ALONG Z-AXIS } SPIN PROJECTION ALONG Z-AXIS IS GOOD QUANTUM NUMBER !

↳ HOW TO INTERPRET THE 2 SOLUTIONS FOR $\vec{p} \neq 0$?

CONSIDER HELICITY OPERATOR $\hat{\Lambda}_S \equiv \frac{\hat{S} \cdot \hat{p}}{|\hat{p}|}$

WITH $\hat{S} = \frac{\hbar}{2} \vec{\Sigma}$

$$\hat{\Lambda}_S = \frac{\hbar}{2} \frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix}$$

COMMUTATOR WITH \hat{H}_D

$$[\hat{H}_D, \hat{\Lambda}_S]$$

$$= \left[c \vec{\alpha} \cdot \hat{\vec{p}} + \beta m_0 c^2, \frac{\hbar}{2} \vec{\Sigma} \cdot \frac{\hat{\vec{p}}}{|\hat{\vec{p}}|} \right]$$

$$= \frac{\hbar c}{2|\hat{\vec{p}}|} \left[\vec{\alpha} \cdot \hat{\vec{p}}, \vec{\Sigma} \cdot \hat{\vec{p}} \right] \quad \left. \begin{array}{l} \text{BECAUSE} \\ \beta \text{ AND } \vec{\Sigma} \text{ ARE} \\ \text{BLOCK DIAGONAL} \\ \text{MATRICES} \end{array} \right\}$$

$$= \frac{\hbar c}{2|\hat{\vec{p}}|} \left\{ \begin{pmatrix} 0 & \vec{\sigma} \cdot \hat{\vec{p}} \\ \vec{\sigma} \cdot \hat{\vec{p}} & 0 \end{pmatrix} \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \cdot \hat{\vec{p}} \\ \vec{\sigma} \cdot \hat{\vec{p}} & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar c}{2|\hat{\vec{p}}|} \left\{ \begin{pmatrix} 0 & (\vec{\sigma} \cdot \hat{\vec{p}})^2 \\ (\vec{\sigma} \cdot \hat{\vec{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\vec{\sigma} \cdot \hat{\vec{p}})^2 \\ (\vec{\sigma} \cdot \hat{\vec{p}})^2 & 0 \end{pmatrix} \right\}$$

$$= 0$$

\hat{H}_D AND $\hat{\Lambda}_S$ CAN BE DIAGONALIZED

SIMULTANEOUSLY



SOLUTIONS OF DIRAC EQUATION

ARE EIGENSTATES OF $\hat{\Lambda}_S$

↳ EIGENVALUE OF $\hat{\Lambda}_S$ (HELICITY OF PARTICLE)

IS A GOOD QUANTUM NUMBER TO CHARACTERIZE SOLUTIONS OF DIRAC EQ.

↳ HELICITY EIGENSTATE FOR PARTICLE SOLUTION

PARTICLE SOLUTION OF DIRAC EQ.

$$\underline{\Psi}(\vec{x}, t) = e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} U_{\epsilon}(\vec{p})$$

WITH $U_{\epsilon}(\vec{p}) = \begin{pmatrix} \psi_{\epsilon} \\ \frac{c \vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \psi_{\epsilon} \end{pmatrix} \quad \epsilon = 1, 2$

↳ $\frac{\hat{S} \cdot \hat{p}}{|\hat{p}|} \underline{\Psi}(\vec{x}, t)$

$$= \frac{\hbar}{2} \frac{1}{|\vec{p}|} \sum_{\epsilon} (-i \vec{v}) e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} U_{\epsilon}(\vec{p})$$

$$= \frac{\hbar}{2} e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} \frac{\sum_{\epsilon} \vec{p}}{|\vec{p}|} U_{\epsilon}(\vec{p})$$

NOTE: \vec{p} (WITHOUT HAT) IS EIGENVALUE OF OPERATOR \hat{p}

↳ $\frac{\sum_{\epsilon} \vec{p}}{|\vec{p}|} U_{\epsilon}(\vec{p}) = \frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} \end{pmatrix} \begin{pmatrix} \psi_{\epsilon} \\ \frac{c \vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \psi_{\epsilon} \end{pmatrix}$

$$= \frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} \psi_{\epsilon} \\ c \frac{(\vec{\sigma} \cdot \vec{p})^2}{E_{\vec{p}} + m_0 c^2} \psi_{\epsilon} \end{pmatrix}$$



CHOOSE φ_κ SUCH THAT

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \varphi_\kappa = \pm \varphi_\kappa \quad \begin{array}{l} + \text{ FOR } \kappa = 1 \\ - \text{ FOR } \kappa = 2 \end{array}$$

\Downarrow

$$\frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} U_\kappa(\vec{p}) = \pm \begin{pmatrix} \varphi_\kappa \\ \frac{c \vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \varphi_\kappa \end{pmatrix} = \pm U_\kappa(\vec{p})$$

\Downarrow

$$\frac{\vec{S} \cdot \vec{p}}{|\vec{p}|} \Psi(\vec{x}, t) = \left(\pm \frac{\hbar}{2} \right) \Psi(\vec{x}, t)$$

$\Psi(\vec{x}, t)$ IS EIGENSTATE OF $\hat{\Lambda}_S$
WITH EIGENVALUE $\pm \frac{\hbar}{2}$

WE DENOTE EIGENVALUE AS $\lambda = \pm \frac{\hbar}{2}$
SPIN PROJECTION ALONG DIRECTION OF MOMENTUM

$$\frac{\vec{S} \cdot \vec{p}}{|\vec{p}|} \Psi(\vec{x}, t) = \lambda \Psi(\vec{x}, t)$$

\hookrightarrow FOR ARBITRARY DIRECTION

$$\frac{\vec{p}}{|\vec{p}|} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

↳ POSITIVE HELICITY STATE ($\lambda = +\frac{\hbar}{2}$)

CORRESPONDS WITH $\kappa = 1$

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \psi_{\lambda = +\frac{\hbar}{2}} = + \psi_{\lambda = +\frac{\hbar}{2}}$$

↳ DENOTE STATE BY ITS EIGENVALUE (OF $\frac{\hbar}{2} \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$)

$$\begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = + \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\psi_{\lambda = +\frac{\hbar}{2}}}$$

$$a \cos \theta + b \sin \theta e^{-i\phi} = a$$

$$b \sin \theta e^{-i\phi} = a 2 \sin^2 \theta/2$$

$$b \cos \theta/2 e^{-i\phi} = a \sin \theta/2$$

$$\left\| \psi_{\lambda = +\frac{\hbar}{2}} \right\rangle = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix}$$

CORRESPONDS TO NORMALIZATION CHOICE

$$\psi_{\lambda = +\frac{\hbar}{2}}^\dagger \psi_{\lambda = +\frac{\hbar}{2}} = 1.$$

↳ NEGATIVE HELICITY STATE ($\lambda = -\frac{\hbar}{2}$)

CORRESPONDS WITH $\kappa = 2$

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \varphi_{\lambda = -\frac{\hbar}{2}} = - \varphi_{\lambda = -\frac{\hbar}{2}}$$

ANALOGOUSLY

$$\left| \varphi_{\lambda = -\frac{\hbar}{2}} \right\rangle = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$$\varphi_{\lambda}^{\dagger} \varphi_{\lambda'} = \delta_{\lambda\lambda'} \quad ; \quad \text{ORTHONORMALITY CONDITION}$$

↳ DIRAC PARTICLE SPINOR SOLUTIONS CAN BE DENOTED AS

$$U(\vec{p}, \lambda) \quad \text{WITH} \quad \lambda = +\frac{\hbar}{2} \leftrightarrow \kappa = 1$$

$$\lambda = -\frac{\hbar}{2} \leftrightarrow \kappa = 2$$

$$\left| U(\vec{p}, \lambda) = N \begin{pmatrix} \varphi_{\lambda} \\ \frac{c \vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \varphi_{\lambda} \end{pmatrix} \right., \quad \text{WITH } \lambda = \pm \frac{\hbar}{2}$$

$$\frac{\hat{S} \cdot \vec{p}}{|\vec{p}|} U(\vec{p}, \lambda) = \lambda U(\vec{p}, \lambda) \quad \text{HELICITY EIGEN STATES}$$

N IS NORMALIZATION FACTOR (SEE FURTHER)

↳ HELICITY EIGENSTATE FOR ANTI-PARTICLE SOLUTION

ANTI-PARTICLE SOLUTION OF DIRAC EQ.

$$\underline{\Psi}(\bar{x}, t) = e^{+\frac{i}{\hbar} \bar{p} \cdot \bar{x}} \underline{u}_{\kappa}(\bar{p})$$

$$\text{WITH } \underline{u}_{\kappa}(\bar{p}) = \begin{pmatrix} \frac{c \bar{\sigma} \cdot \bar{p}}{E_{\bar{p}} + m_0 c^2} \chi_{\kappa} \\ \chi_{\kappa} \end{pmatrix} \quad \text{FOR } \kappa = 1, 2$$

$$\text{↳ } \frac{\hat{S} \cdot \hat{\bar{p}}}{|\bar{p}|} \underline{\Psi}(\bar{x}, t) = -\frac{\hbar}{2} e^{+\frac{i}{\hbar} \bar{p} \cdot \bar{x}} \frac{\sum \cdot \bar{p}}{|\bar{p}|} \underline{u}_{\kappa}(\bar{p})$$

CHOOSE

$$\left\| \frac{\hat{S} \cdot \hat{\bar{p}}}{|\bar{p}|} \underline{u}(\bar{p}, \lambda) = -\lambda \underline{u}(\bar{p}, \lambda) \right.$$

$$\text{WITH } \lambda = +\frac{\hbar}{2}, \quad \kappa = 1$$

$$\lambda = -\frac{\hbar}{2}, \quad \kappa = 2$$

NOTE: FOR ANTI-PARTICLE SPINOR
OPPOSITE SIGN FOR \underline{u} !

↓

$$\frac{\hat{S} \cdot \hat{\bar{p}}}{|\bar{p}|} \underline{\Psi}(\bar{x}, t) = \lambda \underline{\Psi}(\bar{x}, t)$$

$$\sum_{\lambda} \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \psi(\vec{p}, \lambda = \pm \frac{\hbar}{2}) = \mp \psi(\vec{p}, \lambda = \pm \frac{\hbar}{2})$$

$$\frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & \\ & \vec{\sigma} \cdot \vec{p} \end{pmatrix} \begin{pmatrix} c \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \chi_{\pm} \\ \chi_{\pm} \end{pmatrix} = \mp \begin{pmatrix} c \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \chi_{\pm} \\ \chi_{\pm} \end{pmatrix}$$

SATISFIED FOR

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \chi_1 = - \chi_1$$

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \chi_2 = + \chi_2$$

$$\chi_1 = \psi_{\lambda = -\frac{\hbar}{2}} = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$$\chi_2 = \psi_{\lambda = +\frac{\hbar}{2}} = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix}$$

∴ DIRAC ANTI-PARTICLE SPINORS CAN BE DENOTED AS :

$$\psi(\vec{p}, \lambda) = N \begin{pmatrix} c \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m_0 c^2} \psi_{-\lambda} \\ \psi_{-\lambda} \end{pmatrix}$$

$$\hat{S} \cdot \frac{\vec{p}}{|\vec{p}|} \psi(\vec{p}, \lambda) = -\lambda \psi(\vec{p}, \lambda)$$

• NON-RELATIVISTIC CORRESPONDENCE OF DIRAC EQ.

→ CONSIDER NON-REL. PARTICLE ($v/c \ll 1$) WITH CHARGE e MOVING IN APPLIED ELECTROMAGNETIC FIELD $A^\mu (\Phi, \vec{A})$

↓
DESCRIBED BY 'MINIMAL SUBSTITUTION' (SEE ABOVE)

$\hat{p} \rightarrow \hat{p} - \frac{e}{c} \vec{A}$
& ELECTROSTATIC POTENTIAL $e\Phi$

$$\underbrace{\left[c \hat{\alpha} \cdot \left(\hat{p} - \frac{e}{c} \vec{A} \right) + \hat{\beta} m_0 c^2 + e\Phi \right]}_{\hat{H}} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = c \hat{\alpha} \cdot \hat{p} + \hat{\beta} m_0 c^2$$
 FREE DIRAC HAM.

$$\hat{H}_1 = - e \hat{\alpha} \cdot \vec{A} + e\Phi$$

→ NON-REL. REDUCTION OF DIRAC EQ.

$$\underline{\Psi}(\vec{x}, t) = e^{-\frac{i}{\hbar} m_0 c^2 t} \begin{bmatrix} \psi \\ \chi \end{bmatrix}$$

↑

- * FOR POSITIVE ENERGY SOLUTIONS:
- * KINETIC ENERGY APPROXIMATED RELATIVE TO REST MASS

$$i\hbar \frac{\partial \underline{\Psi}}{\partial t} = (m_0 c^2) \underline{\Psi} + e^{-\frac{i}{\hbar} m_0 c^2 t} \left(i\hbar \frac{\partial}{\partial t} \right) \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

⇓

$$\left(c \hat{\alpha} \cdot \hat{\pi} + \beta m_0 c^2 + e \bar{\Phi} \right) \begin{pmatrix} \psi \\ \chi \end{pmatrix} = m_0 c^2 \begin{pmatrix} \psi \\ \chi \end{pmatrix} + i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

WITH $\hat{\pi} \equiv \hat{p} - \frac{e}{c} \bar{A}$

⇓

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = c \begin{pmatrix} \bar{\sigma} \cdot \hat{\pi} & \chi \\ \bar{\sigma} \cdot \hat{\pi} & \psi \end{pmatrix} - 2m_0 c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix} + e \bar{\Phi} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

$$\begin{cases} i\hbar \frac{\partial \psi}{\partial t} = c \bar{\sigma} \cdot \hat{\pi} \chi + e \bar{\Phi} \psi \\ i\hbar \frac{\partial \chi}{\partial t} = c \bar{\sigma} \cdot \hat{\pi} \psi - 2m_0 c^2 \chi + e \bar{\Phi} \chi \end{cases}$$

⇒ 2° EQ. : NEGLECT KINETIC ENERGY $i\hbar \frac{\partial \chi}{\partial t}$ CONTRIBUTION
 & POTENTIAL ENERGY $e \bar{\Phi} \chi$ CONTRIBUTION
 RELATIVE TO REST MASS TERM $- 2m_0 c^2 \chi$

$$\Rightarrow \chi \simeq \frac{\bar{\sigma} \cdot \hat{\pi}}{2m_0 c} \psi$$

→ 1^o EQ. :

$$i\hbar \frac{\partial \psi}{\partial t} \approx c (\vec{\sigma} \cdot \hat{\pi}) \frac{\vec{\sigma} \cdot \hat{\pi}}{2m_0 c} \psi + e\Phi \psi$$

USE $(\vec{\sigma} \cdot \hat{\pi})(\vec{\sigma} \cdot \hat{\pi}) = \hat{\pi} \cdot \hat{\pi} + i \vec{\sigma} \cdot (\hat{\pi} \times \hat{\pi})$

$$\hat{\pi} \cdot \hat{\pi} = \left(\hat{p} - \frac{e}{c} \bar{A} \right) \cdot \left(\hat{p} - \frac{e}{c} \bar{A} \right)$$

$$= \hat{p}^2 - \frac{e}{c} (\hat{p} \cdot \bar{A} + \bar{A} \cdot \hat{p}) + \frac{e^2}{c^2} \bar{A}^2$$

$$\hat{\pi} \times \hat{\pi} = \left(\hat{p} - \frac{e}{c} \bar{A} \right) \times \left(\hat{p} - \frac{e}{c} \bar{A} \right)$$

$$= -\frac{e}{c} (\hat{p} \times \bar{A} + \bar{A} \times \hat{p})$$

$$= i\hbar \frac{e}{c} (\nabla \times \bar{A})$$

$$= i \frac{e\hbar}{c} \bar{B}$$

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{\left(\hat{p} - \frac{e}{c} \bar{A} \right)^2}{2m_0} - \frac{e\hbar}{2m_0 c} \vec{\sigma} \cdot \bar{B} + e\Phi \right] \psi$$

PAULI EQUATION

↳ DESCRIBES NON-REL. PARTICLE OF SPIN 1/2 MOVING IN AN ELECTROMAGNETIC FIELD

2^o TERM : $-\vec{\mu} \cdot \bar{B}$

WITH $\vec{\mu}$: MAGNETIC MOMENT $\vec{\mu} = \frac{e}{2m_0 c} g \vec{S}$

SPIN $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$, $g=2$: GYROMAGNETIC RATIO