Exercise sheet 8
Theoretical Physics 5 : WS 2021/2022
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06.12.2021

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (40 points) : Rotation around x-axis over an angle φ

A finite rotation can be expressed as a succession of N infinitesimal rotations for $N \to \infty$. For a rotation around the x-axis, this can be written as

$$x'^{\mu} = \lim_{N\to\infty} \left[\left(1 + \frac{\varphi}{N} J_x \right)^N \right]^{\mu}_{\ \nu} x^{\nu},$$

where J_x is the generator of an infinitesimal rotation around the x-axis. Show, that the finite rotation is given by

$$x'^{\mu} = \left[1 + J_x^2 + \cos\varphi(-J_x^2) + \sin\varphi J_x \right]^{\mu}_{\ \nu} x^{\nu}$$

and calculate the corresponding rotation matrix.

Exercise 2. (60 points) : Dirac bilinears

Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

- $\Gamma_S = \mathbb{1}$ (scalar);
- $\Gamma_P = \gamma_5$ (pseudoscalar);
- $\Gamma_V^{\mu} = \gamma^{\mu}$ (vector);
- $\Gamma_A^{\mu} = \gamma^{\mu}\gamma_5$ (axial vector).

Without referring to any explicit representation for the Γ matrices,

a) (*10 p.*) show that $\Gamma^2 = \pm\mathbb{1}$;

b) (*15 p.*) show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$; *Hint*: Show first that for any Γ except Γ_S, there always exists a Γ' such that $\{\Gamma, \Gamma'\} = 0$.

c) (*10 p.*) check if the product of two different Γ's is proportional to some Γ different from Γ_S;

d) (*25 p.*) and using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check if the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\mu\gamma_5\psi$.