Exercise sheet 9
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (100 points) :
Dirac particle in a spherical potential well

Consider the Dirac equation

$$\left[\hat{\vec{\alpha}} \cdot \hat{\vec{p}}\,c + \hat{\beta} m_0 c^2\right] \psi(\vec{r}) = [E - V(r)]\psi(\vec{r}),$$

in a spherical potential well:

$$V(r) = \begin{cases} -V_0 & \text{for} \quad r \leq R, \\ 0 & \text{for} \quad r > R. \end{cases}$$

a) (*25 p.*) Show that

$$\hat{\vec{\alpha}} \cdot \hat{\vec{p}} = -i(\hat{\vec{\alpha}} \cdot \vec{e}_r)\left(\hbar\frac{\partial}{\partial r} + \frac{\hbar}{r} - \frac{\beta}{r}\hat{K}\right),$$

with $\hat{K} = \hat{\beta}\left(\hat{\vec{\Sigma}} \cdot \hat{\vec{L}} + \hbar\right)$ and $\vec{e}_r = \vec{r}/r$.
Hint: Use $\vec{\nabla} = \vec{e}_r(\vec{e}_r \cdot \vec{\nabla}) - \vec{e}_r \times (\vec{e}_r \times \vec{\nabla})$.

b) (*25 p.*) Use the ansatz

$$\psi(\vec{r}) = \begin{pmatrix} g(r)\phi_{jl_Am}(\theta, \phi) \\ if(r)\phi_{jl_Bm}(\theta, \phi) \end{pmatrix},$$

where $\phi_{jl_{A/B}m}$ are the eigenstates of the $\hat{\sigma}\hat{L}$ operator

$$\hat{\vec{\sigma}} \cdot \hat{\vec{L}}\phi_{jl_Am} = -\hbar(\kappa + 1)\phi_{jl_Am}$$
$$\hat{\vec{\sigma}} \cdot \hat{\vec{L}}\phi_{jl_Bm} = -\hbar(-\kappa + 1)\phi_{jl_Bm},$$

to find the differential equations for $G(r)$ and $F(r)$ which are related to $g(r), f(r)$ as

$$f(r) = \frac{F(r)}{r}, \qquad g(r) = \frac{G(r)}{r}.$$

c) (*25 p.*) For $k^2 \equiv (\frac{1}{(\hbar c)^2}(E+V_0)^2 - m_0^2c^4) > 0$ the general solution is given by:

$$G(r) = r\left[a_1 j_{l_A}(kr) + a_2 y_{l_A}(kr)\right],$$
$$F(r) = \frac{\kappa}{|\kappa|}\frac{\hbar ckr}{E+V_0+m_0c^2}\left[a_1 j_{l_B}(kr) + a_2 y_{l_B}(kr)\right],$$

where j_l and y_l are the spherical Bessel functions of the first and second kind.
For $\tilde{k}^2 \equiv \frac{1}{(\hbar c)^2}(m_0^2c^4 - (E+V_0)^2) > 0$ the general solution is given by:

$$G(r) = r\sqrt{\frac{2\tilde{k}r}{\pi}}\left[b_1 K_{l_A+1/2}(\tilde{k}r) + b_2 I_{l_A+1/2}(\tilde{k}r)\right],$$

$$F(r) = \frac{\hbar c\tilde{k}r}{E+V_0+m_0c^2}\sqrt{\frac{2\tilde{k}r}{\pi}}\left[-b_1 K_{l_B+1/2}(\tilde{k}r) + b_2 I_{l_B+1/2}(\tilde{k}r)\right],$$

where $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions. Furthermore, it is

$$l_A = \begin{cases} j+\frac{1}{2} & \text{for} \quad \kappa = +(j+\frac{1}{2}), \\ j-\frac{1}{2} & \text{for} \quad \kappa = -(j+\frac{1}{2}) \end{cases}$$

and

$$l_B = \begin{cases} j-\frac{1}{2} & \text{for} \quad \kappa = +(j+\frac{1}{2}), \\ j+\frac{1}{2} & \text{for} \quad \kappa = -(j+\frac{1}{2}). \end{cases}$$

Determine the bound states, for which $E - m_0c^2 > -V_0$, $-m_0c^2 < E < m_0c^2$.

d) (*25 p.*) Exploiting the continuity condition at $r = R$, derive the following relation for S-states ($l = 0$) which correspond to $j = \frac{1}{2}$ and $\kappa = -1$:

$$\frac{kR\sin(kR)}{\sin(kR) - kR\cos(kR)} = \frac{k}{\tilde{k}}\frac{e^{-\tilde{k}R}}{e^{-\tilde{k}R}\left(1+\frac{1}{\tilde{k}R}\right)}\frac{E+m_0c^2}{E+V_0+m_0c^2}.$$

Rewrite the above equation into:

$$\tan\left(\frac{R}{\hbar c}\sqrt{(E+V_0)^2 - m_0^2c^4}\right)\sqrt{\frac{E+V_0+m_0c^2}{E+V_0-m_0c^2}}\times$$
$$\left\{\frac{\hbar c}{R}\left[\frac{1}{E+m_0c^2} - \frac{1}{E+V_0+m_0c^2}\right] - \sqrt{\frac{m_0c^2-E}{m_0c^2+E}}\right\} = 1.$$

These equations relate the energy eigenvalues of the s-states and the properties of the spherical potential well.
Hint: If necessary, you can use the following Rayleigh formulas for (spherical) Bessel functions:

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin(x)}{x}\right),$$
$$y_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos(x)}{x}\right),$$
$$i_n(x) = x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sinh(x)}{x}\right),$$
$$k_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{e^{-x}}{x}\right),$$

with

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x) \qquad \text{and} \qquad k_n(x) = \sqrt{\frac{\pi}{2x}} K_{n+1/2}(x).$$